

ON FUNCTIONAL EQUATIONS OF FINITE MULTIPLE POLYLOGARITHMS

KENJI SAKUGAWA AND SHIN-ICHIRO SEKI

ABSTRACT. Recently, several people study finite multiple zeta values (FMZVs) and finite polylogarithms (FPs). In this paper, we introduce finite multiple polylogarithms (FMPs), which are natural generalizations of FMZVs and FPs, and we establish functional equations of FMPs. As applications of these functional equations, we calculate special values of FMPs containing generalizations of congruences obtained by Meštrović, Z. W. Sun, L. L. Zhao, Tauraso, and J. Zhao. We show supercongruences for certain generalized Bernoulli numbers and the Bernoulli numbers as an appendix.

1. INTRODUCTION

From the end of twentieth century to the beginning of twenty-first century, Hoffman and J. Zhao had started research about mod p multiple harmonic sums, which are motivated by various generalizations of classical Wolstenholme's theorem. Recently, Kaneko and Zagier introduced a new “adélic” framework to describe the pioneer works by Hoffman and Zhao and they defined finite multiple zeta values (FMZVs). Let k_1, \dots, k_m be positive integers and $\mathbb{k} := (k_1, \dots, k_m)$.

Definition 1.1 (Kaneko and Zagier [10, 11]). *The finite multiple zeta value $\zeta_{\mathcal{A}}(\mathbb{k})$ is defined by*

$$\zeta_{\mathcal{A}}(\mathbb{k}) := \left(\sum_{p > n_1 > \dots > n_m > 0} \frac{1}{n_1^{k_1} \dots n_m^{k_m}} \bmod p \right)_p \in \mathcal{A}$$

and the finite multiple zeta-star value $\zeta_{\mathcal{A}}^*(\mathbb{k})$ is defined by

$$\zeta_{\mathcal{A}}^*(\mathbb{k}) := \left(\sum_{p-1 \geq n_1 \geq \dots \geq n_m \geq 1} \frac{1}{n_1^{k_1} \dots n_m^{k_m}} \bmod p \right)_p \in \mathcal{A}.$$

Here, the \mathbb{Q} -algebra \mathcal{A} is defined by

$$\mathcal{A} := \left(\prod_p \mathbb{F}_p \right) / \left(\bigoplus_p \mathbb{F}_p \right),$$

where p runs over all prime numbers.

In this framework, Kaneko and Zagier established a conjecture, which states that there is an isomorphism between the \mathbb{Q} -algebra spanned by FMZVs and the quotient \mathbb{Q} -algebra modulo the ideal generated by $\zeta(2)$ of the \mathbb{Q} -algebra spanned by the usual multiple zeta values.

On the other hand, Kontsevich [14], Elbaz-Vincent and Gangl [4] introduced finite version of polylogarithms and studied functional equations of them. Based on their works, Mattarei and Tauraso [19] calculated special values of finite polylogarithms.

Inspired by these studies, we introduce a finite version of multiple polylogarithms in the framework of Kaneko and Zagier:

Definition 1.2 (See Definition 3.8). *The finite multiple polylogarithms (FMPs) $\mathfrak{L}_{\mathcal{A},\mathbb{k}}(t)$, $\mathfrak{L}_{\mathcal{A},\mathbb{k}}^*(t)$, $\tilde{\mathfrak{L}}_{\mathcal{A},\mathbb{k}}(t)$, and $\tilde{\mathfrak{L}}_{\mathcal{A},\mathbb{k}}^*(t)$ are defined by*

$$\begin{aligned}\mathfrak{L}_{\mathcal{A},\mathbb{k}}(t) &:= \left(\sum_{p > n_1 > \dots > n_m > 0} \frac{t^{n_1}}{n_1^{k_1} \dots n_m^{k_m}} \bmod p \right)_p \in \mathcal{A}_{\mathbb{Z}[t]}, \\ \mathfrak{L}_{\mathcal{A},\mathbb{k}}^*(t) &:= \left(\sum_{p-1 \geq n_1 \geq \dots \geq n_m \geq 1} \frac{t^{n_1}}{n_1^{k_1} \dots n_m^{k_m}} \bmod p \right)_p \in \mathcal{A}_{\mathbb{Z}[t]}, \\ \tilde{\mathfrak{L}}_{\mathcal{A},\mathbb{k}}(t) &:= \left(\sum_{p > n_1 > \dots > n_m > 0} \frac{t^{n_m}}{n_1^{k_1} \dots n_m^{k_m}} \bmod p \right)_p \in \mathcal{A}_{\mathbb{Z}[t]}, \\ \tilde{\mathfrak{L}}_{\mathcal{A},\mathbb{k}}^*(t) &:= \left(\sum_{p-1 \geq n_1 \geq \dots \geq n_m \geq 1} \frac{t^{n_m}}{n_1^{k_1} \dots n_m^{k_m}} \bmod p \right)_p \in \mathcal{A}_{\mathbb{Z}[t]}.\end{aligned}$$

Here, the \mathbb{Q} -algebra $\mathcal{A}_{\mathbb{Z}[t]}$ is defined by

$$\mathcal{A}_{\mathbb{Z}[t]} = \left(\prod_p \mathbb{F}_p[t] \right) / \left(\bigoplus_p \mathbb{F}_p[t] \right),$$

where p runs over all prime numbers.

The symbol \mathfrak{L} is used for the finite polylogarithms by Elbaz-Vincent and Gangl in their paper [4].

The main purpose of this paper is to establish functional equations of FMPs. Special cases of the main results are as follows:

Theorem 1.3 (Main Theorem). *The following functional equations hold in $\mathcal{A}_{\mathbb{Z}[t]}$:*

$$\begin{aligned}(1) \quad & \tilde{\mathfrak{L}}_{\mathcal{A},\mathbb{k}}^*(t) = \tilde{\mathfrak{L}}_{\mathcal{A},\mathbb{k}^\vee}^*(1-t) - \zeta_{\mathcal{A}}^*(\mathbb{k}^\vee), \\ (2) \quad & (-1)^{m-1} \mathfrak{L}_{\mathcal{A},\mathbb{k}}(t) = \tilde{\mathfrak{L}}_{\mathcal{A},\bar{\mathbb{k}}}^*(t) + \sum_{j=1}^{m-1} (-1)^j \mathfrak{L}_{\mathcal{A},(k_1, \dots, k_j)}(t) \zeta_{\mathcal{A}}^*(k_m, \dots, k_{j+1}).\end{aligned}$$

Here, $\bar{\mathbb{k}}$ is the reverse index and \mathbb{k}^\vee the Hoffman dual of \mathbb{k} (see Subsection 2.1).

The equality (1) is a generalization of the Hoffman duality [8, Theorem 4.6]. The precise version of the main results are Theorem 3.12, Corollary 3.13, Remark 3.14, and Theorem 3.15 which consist of the multiple variable cases of (1) and (2), the \mathcal{A}_2 -version of (1), and the \mathcal{A}_n -version of (2) for any positive integer n (see Theorem 1.5 below and Subsection 3.1 for the definition of \mathcal{A}_n). Main results are obtained as mod p reductions of generalizations (= Theorem 2.5 and Theorem 2.10) of classical Euler's identity:

$$(3) \quad \sum_{n=1}^N (-1)^{n-1} \binom{N}{n} \frac{1}{n} = \sum_{n=1}^N \frac{1}{n},$$

where N is a positive integer ([5]). The functional equation (2) and its generalization also hold for the usual multiple polylogarithms (Theorem 2.13).

As applications of the functional equations, we will calculate some special values of the finite multiple polylogarithms by using Tauraso and J. Zhao's results for the alternating multiple harmonic sums in Subsection 4.1.

Several people study supercongruences involving the harmonic numbers (Z. W. Sun, L. L. Zhao, Meštrović, and so on). For instance, Z. W. Sun and L. L. Zhao proved the following congruence:

Theorem 1.4 (Z. W. Sun and L. L. Zhao [26, Theorem 1.1]). *Let p be a prime number greater than 3. Then*

$$\sum_{k=1}^{p-1} \frac{H_k}{k2^k} \equiv \frac{7}{24} p B_{p-3} \pmod{p^2},$$

where $H_k = \sum_{j=1}^k 1/j$ is the k -th harmonic number and B_{p-3} is the $(p-3)$ -rd Bernoulli number.

We can regard such congruences as explicit formulas of special values of FMPs and we will give generalizations of some of them. As an application of main results, we obtain the following theorem which is a generalization of Theorem 1.4 (= the case $m = 2$ in Theorem 1.5):

Theorem 1.5 (cf. Theorem 4.6). *Let m be an even positive integer. Then we have*

$$\mathfrak{L}_{\mathcal{A}_2, \{1\}^m}^*(1/2) = \left(\frac{2^{m+1} - 1}{2^{m+1}} \frac{B_{p-m-1}}{m+1} p \pmod{p^2} \right)_p \text{ in } \mathcal{A}_2,$$

where B_{p-m-1} is the $(p-m-1)$ -st Bernoulli number and

$$\mathfrak{L}_{\mathcal{A}_2, \{1\}^m}^*(1/2) := \left(\sum_{p-1 \geq n_1 \geq \dots \geq n_m \geq 1} \frac{1}{n_1 \cdots n_m 2^{n_1}} \pmod{p^2} \right)_p \text{ in } \mathcal{A}_2.$$

Here, the \mathbb{Q} -algebra \mathcal{A}_2 is defined by

$$\mathcal{A}_2 := \left(\prod_p \mathbb{Z}/p^2\mathbb{Z} \right) \bigg/ \left(\bigoplus_p \mathbb{Z}/p^2\mathbb{Z} \right).$$

Independently of us, Ono and Yamamoto gave another definition of FMPs and established the shuffle relation in their preprint [21]. We will investigate a relation between our FMPs and Ono-Yamamoto's FMPs and calculate its special values.

This paper is organized as follows:

In Section 2, we prove generalizations of Euler's identity involving binomial coefficients by introducing some formal truncated integral operators. In Section 3, we define the ring $\mathcal{A}_{n,R}^\Sigma$ and recall some known results on FMZVs. Then we introduce FMPs and prove the functional equations by applying identities obtained in Section 2. In Section 4, as applications of the functional equations, we calculate special values of FMPs. We also study the relation between our FMPs and the ones defined by Ono and Yamamoto. There are two appendices in this paper. In A, we give an elementary proof of supercongruences for the generalized Bernoulli number for powers of the Teichmüller character and the Bernoulli numbers. B is a table of sufficient conditions for special values of FMPs.

2. GENERALIZATIONS OF EULER'S IDENTITY

2.1. Notations for indices and the Hoffman dual. Here, we recall an involution introduced by Hoffman on the set of indices.

We define the set I by

$$I := \coprod_{m \in \mathbb{Z}_{>0}} \underbrace{(\mathbb{Z}_{>0} \times \cdots \times \mathbb{Z}_{>0})}_m$$

and we call an element of I an *index*. For an index $\mathbb{k} = (k_1, \dots, k_m) \in I$ we define *the weight* (resp. *the depth*) of \mathbb{k} to be $k_1 + \cdots + k_m$ (resp. m) and we denote it by $\text{wt}(\mathbb{k})$ (resp. $\text{dep}(\mathbb{k})$).

For a non-negative integer k , the symbol $\{k\}^m$ denotes m repetitions $(k, \dots, k) \in \mathbb{Z}_{\geq 0}^m$ of k . Let $\mathbf{e}_i := (\{0\}^{i-1}, 1, \{0\}^{m-i})$ when m is clear from the context. Throughout this paper, we use the following operators for indices:

For three indices $\mathbb{k} = (k_1, \dots, k_m)$, $\mathbb{k}_1 = (k'_1, \dots, k'_m)$, and $\mathbb{k}_2 = (k''_1, \dots, k''_{m'})$, we define $\overline{\mathbb{k}}$, $\mathbb{k}_1 \sqcup \mathbb{k}_2$, and $\mathbb{k} \oplus \mathbb{k}_1$ by $\overline{\mathbb{k}} := (k_m, \dots, k_1)$, $\mathbb{k}_1 \sqcup \mathbb{k}_2 := (k'_1, \dots, k'_m, k''_1, \dots, k''_{m'})$, and $\mathbb{k} \oplus \mathbb{k}_1 := (k_1 + k'_1, \dots, k_m + k'_m)$, respectively.

Let W be the free monoid generated by the set $\{0, 1\}$. We denote by W_1 the set of words in W of the form $\cdots 1$. We see that the correspondence

$$(k_1, \dots, k_m) \mapsto \underbrace{0 \cdots 0 1}_{k_1-1} \underbrace{0 \cdots 0 1}_{k_2-1} \cdots 1 \underbrace{0 \cdots 0 1}_{k_m-1}$$

induces a bijection $w: I \xrightarrow{\sim} W_1$.

Definition 2.1 (cf. [8, Section 3]). Let $\tau: W \xrightarrow{\sim} W$ be a monoid homomorphism defined by $\tau(0) = 1$ and $\tau(1) = 0$. Then we define an involution $^\vee: I \rightarrow I$ by the equality $w(\mathbb{k}^\vee) = \tau(w(\mathbb{k})1^{-1})1$. We call this involution *the Hoffman dual*.

By the definition of the Hoffman dual, we see that $\mathbb{k}^{\vee\vee} = \mathbb{k}$ holds for any index \mathbb{k} . We can use the notation $\overline{\mathbb{k}}^\vee$ since the Hoffman dual and the reversal operator $\mathbb{k} \mapsto \overline{\mathbb{k}}$ commute.

Example 2.2. We have the following equalities:

$$\begin{aligned} m^\vee &= \{1\}^m, \quad (k_1, k_2)^\vee = (\{1\}^{k_1-1}, 2, \{1\}^{k_2-1}), \\ (k_1, k_2, k_3)^\vee &= (\{1\}^{k_1-1}, 2, \{1\}^{k_2-2}, 2, \{1\}^{k_3-1}), \\ (k_1, \{1\}^{k_2-1})^\vee &= (\{1\}^{k_1-1}, k_2). \end{aligned}$$

Here, m, k_1, k_2 and k_3 are positive integers and the third equality holds when k_2 is greater than or equal to 2.

The following lemma is useful for inductive arguments on weight:

Lemma 2.3. *Let \mathbb{k} be an index and \mathbb{k}^\vee its dual. Then we have $(\mathbb{k} \oplus \mathbf{e}_1)^\vee = \{1\} \sqcup \mathbb{k}^\vee$ and $(\{1\} \sqcup \mathbb{k})^\vee = (\mathbb{k}^\vee \oplus \mathbf{e}_1)$.*

Proof. These are obvious by the definition of the Hoffman dual. \square

We can prove that the equalities $\text{wt}(\mathbb{k}^\vee) = \text{wt}(\mathbb{k})$ and $\text{dep}(\mathbb{k}) + \text{dep}(\mathbb{k}^\vee) = \text{wt}(\mathbb{k}) + 1$ hold for any index \mathbb{k} by using the above lemma.

Remark 2.4. Hoffman defined the dual $\mathbb{k} \mapsto \mathbb{k}^\vee$ by another way (see [8, Section 3]). Let $I_w := \{\mathbb{k} \in I \mid \text{wt}(\mathbb{k}) = w\}$ for a positive integer w and $\mathcal{P}(\{1, 2, \dots, w-1\})$ the power set

of $\{1, 2, \dots, w-1\}$. Then there is a bijection $\psi: I_w \xrightarrow{\sim} \mathcal{P}(\{1, 2, \dots, w-1\})$ defined by the correspondence

$$\mathbb{k} = (k_1, \dots, k_m) \mapsto \{k_1, k_1 + k_2, \dots, k_1 + \dots + k_{m-1}\}$$

and the original Hoffman dual \mathbb{k}^\vee is defined by

$$\mathbb{k}^\vee := \psi^{-1}(\{1, 2, \dots, w-1\} \setminus \psi(\mathbb{k}))$$

for $\mathbb{k} \in I_w$. This definition is equivalent to the first definition since the identities in Lemma 2.3 also hold for this dual.

2.2. Generalizations of Euler's identity and their corollaries. In this subsection, we state polynomial identities which will be used for the proof of our main results in Subsection 3.2. We will give the proofs of Theorem 2.5 and Theorem 2.10 in Subsection 2.4. Through this subsection, let R be a commutative ring including the field of rational numbers.

Theorem 2.5. *Let $\mathbb{k} = (k_1, \dots, k_m)$ be an index of weight w and N a positive integer. Then the following polynomial identity holds in $R[t_1, \dots, t_m]$:*

$$(4) \quad \sum_{N \geq n_1 \geq \dots \geq n_m \geq 1} (-1)^{n_1} \binom{N}{n_1} \frac{t_1^{n_1-n_2} \dots t_{m-1}^{n_{m-1}-n_m} t_m^{n_m}}{n_1^{k_1} \dots n_m^{k_m}} = \sum_{N \geq n_1 \geq \dots \geq n_w \geq 1} \frac{(1-t_1)^{n_{l_1}-n_{l_1+1}} \dots (1-t_{m-1})^{n_{l_{m-1}}-n_{l_{m-1}+1}} \{(1-t_m)^{n_{l_m}} - 1\}}{n_1 \dots n_w},$$

$$(5) \quad \sum_{N \geq n_1 \geq \dots \geq n_m \geq 1} \frac{t_1^{n_1-n_2} \dots t_{m-1}^{n_{m-1}-n_m} t_m^{n_m}}{n_1^{k_1} \dots n_m^{k_m}} = \sum_{N \geq n_1 \geq \dots \geq n_w \geq 1} (-1)^{n_1} \binom{N}{n_1} \frac{(1-t_1)^{n_{l_1}-n_{l_1+1}} \dots (1-t_{m-1})^{n_{l_{m-1}}-n_{l_{m-1}+1}} \{(1-t_m)^{n_{l_m}} - 1\}}{n_1 \dots n_w},$$

where $l_1 = k_1, l_2 = k_1 + k_2, \dots, l_m = k_1 + \dots + k_m (= w)$.

When we substitute 1 for some of t_1, \dots, t_{m-1} in the left hand side of (4), some terms vanish and some 0^0 appear in the summation of the right hand side. We consider 0^0 as 1 since $t^{n-n} = t^0$ is 1 as a constant function for a variable t and a positive integer n . In such situations, we can rewrite (4) as follows:

Corollary 2.6. *We use the same notations as in Theorem 2.5. Let $\{i_1, \dots, i_h\}$ be any subset of $\{1, \dots, w-1\}$ with the complement $\{j_1, \dots, j_{h'}\}$ and put $(k'_1, \dots, k'_{m'}) := (k_1 + \dots + k_{i_1}, k_{i_1+1} + \dots + k_{i_2}, \dots, k_{i_h+1} + \dots + k_m)^\vee$. Then we have*

$$\sum_{N \geq n_1 \geq \dots \geq n_m \geq 1} (-1)^{n_1} \binom{N}{n_1} \frac{t_1^{n_1-n_2} \dots t_{m-1}^{n_{m-1}-n_m} t_m^{n_m}}{n_1^{k_1} \dots n_m^{k_m}} \Big|_{t_{i_1}=\dots=t_{i_h}=1} = \sum_{N \geq n_1 \geq \dots \geq n_{m'} \geq 1} \frac{(1-t_{j_1})^{n_{L_1}-n_{L_1+1}} (1-t_{j_2})^{n_{L_2}-n_{L_2+1}} \dots (1-t_{j_{h'}})^{n_{L_{h'}}-n_{L_{h'}+1}} \{(1-t_m)^{n_{m'}} - 1\}}{n_1^{k'_1} \dots n_{m'}^{k'_{m'}}},$$

where $L_i := l_{j_i} - j_i + i$ for $i = 1, \dots, m'$.

Proof. If the condition

$$(*) \quad n_{l_{i_1}} = n_{l_{i_1}+1}, \quad n_{l_{i_2}} = n_{l_{i_2}+1}, \quad \dots, \quad n_{l_{i_h}} = n_{l_{i_h}+1}$$

holds, then we have

$$\begin{aligned} & (1-t_1)^{n_{l_1}-n_{l_1}+1} (1-t_2)^{n_{l_2}-n_{l_2}+1} \dots (1-t_{m-1})^{n_{l_{m-1}}-n_{l_{m-1}}+1} \{(1-t_m)^{n_{l_m}} - 1\} \\ &= (1-t_{j_1})^{n_{l_{j_1}}-n_{l_{j_1}}+1} (1-t_{j_2})^{n_{l_{j_2}}-n_{l_{j_2}}+1} \dots (1-t_{j_h})^{n_{l_{j_h}}-n_{l_{j_h}}+1} \{(1-t_m)^{n_w} - 1\} \end{aligned}$$

as a polynomial equality. On the other hand, if (n_1, \dots, n_w) does not satisfy the condition $(*)$, then the term for (n_1, \dots, n_w) of the right hand side of Theorem 2.5 (4) vanishes when $t_{i_1} = \dots = t_{i_h} = 1$.

By the definition of the Hoffman dual, we have

$$\{l'_1, l'_2, \dots, l'_{m'-1}\} = \{1, \dots, w-1\} \setminus \{l_{i_1}, \dots, l_{i_h}\}$$

where $l'_1 = k'_1, l'_2 = k'_1 + k'_2, \dots, l'_{m'-1} = k'_1 + \dots + k'_{m'-1}$. Therefore we can rewrite the condition $(*)$ as follows:

$$n_1 = \dots = n_{l'_1}, \quad n_{l'_1+1} = \dots = n_{l'_2}, \quad \dots, \quad n_{l'_{m'-1}+1} = \dots = n_w.$$

Hence we have the desired formula. \square

In particular, we have the following corollary:

Corollary 2.7. *Let $\mathbb{k} = (k_1, \dots, k_m)$ be an index, $\mathbb{k}^\vee = (k'_1, \dots, k'_{m'})$, and N a positive integer. Then we have the polynomial identity*

$$(6) \quad \sum_{N \geq n_1 \geq \dots \geq n_m \geq 1} (-1)^{n_1} \binom{N}{n_1} \frac{t^{n_m}}{n_1^{k_1} \dots n_m^{k_m}} = \sum_{N \geq n_1 \geq \dots \geq n_{m'} \geq 1} \frac{(1-t)^{n_{m'}} - 1}{n_1^{k'_1} \dots n_{m'}^{k'_{m'}}},$$

in $R[t]$.

Remark 2.8. The case $\mathbb{k} = m \in \mathbb{Z}_{>0}$ of Corollary 2.7 (6) gives Tauraso-Zhao's identity [28, Lemam 5.5 (42)] and the case $t = 1$ of Corollary 2.7 (6) gives Hoffman's identity [8, Theorem 4.2]. Dilcher's identity [2] and Hernández's identity [6] are special cases of Hoffman's identity. All these are generalizations of Euler's identity (3).

Remark 2.9. Theorem 2.5 is also deduced from Kawashima-Tanaka's formula [13, Theorem 2.6], which is a generalization of the identity

$$\sum_{n=0}^N (-1)^n \binom{N}{n} \frac{1}{n+1} = \frac{1}{N+1}$$

(cf. [6, Woord's solution]). Our proof of Theorem 2.5 in Subsection 2.4 is quite different from the proof by Kawashima and Tanaka.

Theorem 2.10. Let $\mathbb{k} = (k_1, \dots, k_m)$ be an index of weight w and N a positive integer. Then the following identity holds in $R[t_1, t_2^{\pm 1}, \dots, t_m^{\pm 1}]$:

$$\begin{aligned}
 & \sum_{N+1 > n_1 > \dots > n_m > 0} (-1)^{n_m} \binom{N}{n_m} \frac{(t_1/t_2)^{n_1} \dots (t_{m-1}/t_m)^{n_{m-1}} t_m^{n_m}}{n_1^{k_1} \dots n_m^{k_m}} \\
 &= (-1)^{m-1} \sum_{N \geq n_1 \geq \dots \geq n_w \geq 1} \frac{(1-t_m)^{n_{l_1}-n_{l_1+1}} \dots (1-t_2)^{n_{l_{m-1}}-n_{l_{m-1}+1}} \{(1-t_1)^{n_{l_m}} - 1\}}{n_1 \dots n_w} \\
 (7) \quad &+ \sum_{j=1}^{m-1} (-1)^{m-j-1} \left(\sum_{N+1 > n_1 > \dots > n_j > 0} \frac{(t_1/t_2)^{n_1} \dots (t_j/t_{j+1})^{n_j}}{n_1^{k_1} \dots n_j^{k_j}} \right) \times \\
 &\left(\sum_{N \geq n_1 \geq \dots \geq n_{l_{m-j}} \geq 1} \frac{(1-t_m)^{n_{l_1}-n_{l_1+1}} \dots (1-t_{j+2})^{n_{l_{m-j-1}}-n_{l_{m-j-1}+1}} \{(1-t_{j+1})^{n_{l_{m-j}}} - 1\}}{n_1 \dots n_{l_{m-j}}} \right)
 \end{aligned}$$

and

$$\begin{aligned}
 & \sum_{N+1 > n_1 > \dots > n_m > 0} \frac{(t_1/t_2)^{n_1} \dots (t_{m-1}/t_m)^{n_{m-1}} t_m^{n_m}}{n_1^{k_1} \dots n_m^{k_m}} \\
 &= (-1)^{m-1} \sum_{N \geq n_1 \geq \dots \geq n_w \geq 1} (-1)^{n_1} \binom{N}{n_1} \frac{(1-t_m)^{n_{l_1}-n_{l_1+1}} \dots (1-t_2)^{n_{l_{m-1}}-n_{l_{m-1}+1}} \{(1-t_1)^{n_{l_m}} - 1\}}{n_1 \dots n_w} \\
 (8) \quad &+ \sum_{j=1}^{m-1} (-1)^{m-j-1} \left(\sum_{N+1 > n_1 > \dots > n_j > 0} \frac{(t_1/t_2)^{n_1} \dots (t_j/t_{j+1})^{n_j}}{n_1^{k_1} \dots n_j^{k_j}} \right) \times \\
 &\left(\sum_{N \geq n_1 \geq \dots \geq n_{l_{m-j}} \geq 1} (-1)^{n_1} \binom{N}{n_1} \frac{(1-t_m)^{n_{l_1}-n_{l_1+1}} \dots (1-t_{j+2})^{n_{l_{m-j-1}}-n_{l_{m-j-1}+1}} \{(1-t_{j+1})^{n_{l_{m-j}}} - 1\}}{n_1 \dots n_{l_{m-j}}} \right),
 \end{aligned}$$

where $l_1 = k_m, l_2 = k_m + k_{m-1}, \dots, l_m = k_m + \dots + k_1 (= w)$.

Theorem 2.11. Let $\mathbb{k} = (k_1, \dots, k_m)$ be an index and N a positive integer. Then the following identity holds in $R[t_1, \dots, t_m]$:

$$\begin{aligned}
 & (-1)^{m-1} \sum_{N+1 > n_1 > \dots > n_m > 0} \frac{t_1^{n_1} \dots t_m^{n_m}}{n_1^{k_1} \dots n_m^{k_m}} = \sum_{N \geq n_1 \geq \dots \geq n_m \geq 1} \frac{t_m^{n_1} \dots t_1^{n_m}}{n_1^{k_m} \dots n_m^{k_1}} \\
 &+ \sum_{j=1}^{m-1} (-1)^j \left(\sum_{N+1 > n_1 > \dots > n_j > 0} \frac{t_1^{n_1} \dots t_j^{n_j}}{n_1^{k_1} \dots n_j^{k_j}} \right) \left(\sum_{N \geq n_1 \geq \dots \geq n_{m-j} \geq 1} \frac{t_m^{n_1} \dots t_{j+1}^{n_{m-j}}}{n_1^{k_m} \dots n_{m-j}^{k_{j+1}}} \right).
 \end{aligned}$$

Proof. By combining Theorem 2.5 (5) and Theorem 2.10 (8), we have

$$\begin{aligned}
 & (-1)^{m-1} \sum_{N+1 > n_1 > \dots > n_m > 0} \frac{(t_1/t_2)^{n_1} \dots (t_{m-1}/t_m)^{n_{m-1}} t_m^{n_m}}{n_1^{k_1} \dots n_m^{k_m}} = \sum_{N \geq n_1 \geq \dots \geq n_m \geq 1} \frac{t_m^{n_1-n_2} \dots t_2^{n_{m-1}-n_m} t_1^{n_m}}{n_1^{k_m} \dots n_m^{k_1}} \\
 &+ \sum_{j=1}^{m-1} (-1)^j \left(\sum_{N+1 > n_1 > \dots > n_j > 0} \frac{(t_1/t_2)^{n_1} \dots (t_j/t_{j+1})^{n_j}}{n_1^{k_1} \dots n_j^{k_j}} \right) \left(\sum_{N \geq n_1 \geq \dots \geq n_{m-j} \geq 1} \frac{t_m^{n_1-n_2} \dots t_{j+2}^{n_{m-j-1}-n_{m-j}} t_{j+1}^{n_{m-j}}}{n_1^{k_m} \dots n_{m-j}^{k_{j+1}}} \right)
 \end{aligned}$$

in $R[t_1, t_2^{\pm 1}, \dots, t_m^{\pm 1}]$. By replacing $t_1/t_2 \mapsto t_1, \dots, t_{m-1}/t_m \mapsto t_{m-1}$, we obtain the desired identity. \square

As applications of Theorem 2.11, we can prove not only a formula for the finite multiple polylogarithms (= Theorem 3.15) but also a formula for the usual multiple polylogarithms by taking the limit $N \rightarrow \infty$. We recall the definition of the multiple polylogarithms:

Definition 2.12. Let $\mathbb{k} = (k_1, \dots, k_m)$ be an index and z_1, \dots, z_m complex numbers satisfying at least one of the following conditions for absolute convergence:

- (i) $|z_1| < 1$ and $|z_i| \leq 1$ ($2 \leq i \leq m$), (ii) $|z_i| \leq 1$ ($1 \leq i \leq m$) and $k_1 \geq 2$.

Then, we define *the multiple polylogarithms* by

$$\mathrm{Li}_{\mathbb{k}}(z_1, \dots, z_m) := \sum_{n_1 > n_2 > \dots > n_m \geq 1} \frac{z_1^{n_1} \dots z_m^{n_m}}{n_1^{k_1} \dots n_m^{k_m}}, \quad \mathrm{Li}_{\mathbb{k}}^*(z_1, \dots, z_m) := \sum_{n_1 \geq n_2 \geq \dots \geq n_m \geq 1} \frac{z_1^{n_1} \dots z_m^{n_m}}{n_1^{k_1} \dots n_m^{k_m}}.$$

If $k_1 \geq 2$, then we define *the multiple zeta(-star) values* $\zeta(\mathbb{k})$ and $\zeta^*(\mathbb{k})$ by $\zeta(\mathbb{k}) := \mathrm{Li}_{\mathbb{k}}(\{1\}^m)$ and $\zeta^*(\mathbb{k}) := \mathrm{Li}_{\mathbb{k}}^*(\{1\}^m)$, respectively.

Theorem 2.13. Let $\mathbb{k} = (k_1, \dots, k_m)$ be an index and z_1, \dots, z_m complex numbers satisfying at least one of the following conditions:

- (i) $|z_1| < 1$, $|z_i| \leq 1$ ($2 \leq i \leq m-1$), and $|z_m| < 1$,
(ii) $|z_i| \leq 1$ ($1 \leq i \leq m$), $k_1 \geq 2$, and $k_m \geq 2$.

Then we have

$$\sum_{j=0}^m (-1)^j \mathrm{Li}_{(k_1, \dots, k_j)}(z_1, \dots, z_j) \mathrm{Li}_{(k_m, \dots, k_{j+1})}^*(z_m, \dots, z_{j+1}) = 0.$$

Here, we consider $\mathrm{Li}_{(k_1, \dots, k_j)}(z_1, \dots, z_j)$ (resp. $\mathrm{Li}_{(k_m, \dots, k_{j+1})}^*(z_m, \dots, z_{j+1})$) as 1 when $j = 0$ (resp. $j = m$).

If $k_1 \geq 2$ and $k_m \geq 2$, then we have

$$(9) \quad \sum_{j=0}^m (-1)^j \zeta(k_1, \dots, k_j) \zeta^*(k_m, \dots, k_{j+1}) = 0.$$

See Remark 3.17.

2.3. Truncated integral operators. In this subsection, we introduce *truncated integral operators* to prove the theorems in Subsection 2.2. Through this subsection, let R be a commutative ring including the field of rational numbers \mathbb{Q} and N a positive integer. Let t and s be indeterminates.

Let $\int * dt: R[[t]] \rightarrow R[[t]]$ be the formal indefinite integral operator satisfying the condition that the constant term with respect to t is equal to 0, that is,

$$\int \left(\sum_{n=0}^{\infty} a_n t^n \right) dt := \sum_{n=0}^{\infty} \frac{a_n}{n+1} t^{n+1}.$$

We prepare the following five R -linear operators:

$$\begin{aligned}
I_{t,R}: tR[t] &\longrightarrow tR[t] & I_{t,s;R}: R[t, s] &\longrightarrow R[[s/t]][t] & \tau_{t;R}^{\leq N}: R[[t]] &\longrightarrow R[t] \\
f(t) &\longmapsto \int \frac{f(t)}{t} dt, & f(t, s) &\longmapsto \int \frac{f(t, s)}{t-s} ds, & \sum_{n=0}^{\infty} a_n t^n &\longmapsto \sum_{n=0}^N a_n t^n, \\
\text{pr}_{t;R}: R((t^{-1})) &\longrightarrow R[t] & \text{pr}_{t;R}^-: R((t^{-1})) &\longrightarrow t^{-1}R[[t^{-1}]] \\
\sum_{n=-\infty}^{n_0} a_n t^n &\longmapsto \begin{cases} \sum_{n=0}^{n_0} a_n t^n & \text{if } n_0 \text{ is non-negative,} \\ 0 & \text{otherwise,} \end{cases} & \sum_{n=n_0}^{\infty} a_n t^{-n} &\longmapsto \sum_{n=1}^{\infty} a_n t^{-n}.
\end{aligned}$$

Here, we consider the formal integral operator in the definition of $I_{t,s;R}$ as an operator on $R((t^{-1}))[s]$. For instance, we have

$$(10) \quad I_{t,s;R}(s^n) = \sum_{j=1}^{\infty} \frac{s^{n+j} t^{-j}}{j}$$

for a non-negative integer n .

Definition 2.14. We define the truncated integral operators $J_{t,s;R}^{\star}$ and $J_{t,s;R}^N$ by

$$\begin{aligned}
J_{t,s;R}^{\star} &:= \text{pr}_{t;R[[s]]} \circ I_{t,s;R}: R[t, s] \longrightarrow R[[s]][t], \\
J_{t,s;R}^N &:= \tau_{s;R[[t^{-1}]]}^{\leq N} \circ \text{pr}_{t;R[[s]]}^- \circ I_{t,s;R}: R[t, s] \longrightarrow t^{-1}R[[t^{-1}]] [s].
\end{aligned}$$

We can check easily that the image of $J_{t,s;R}^{\star}$ (resp. $J_{t,s;R}^N$) is included in $R[t, s]$ (resp. $t^{-1}R[t^{-1}, s]$).

For simplicity, we omit the ring R from our notations. The following Lemma 2.15 and Lemma 2.18 are fundamental for the proofs of Theorem 2.5 and Theorem 2.10.

Lemma 2.15. Let n be a positive integer. Then we have the following identities:

$$(11) \quad I_t(t^n) = \frac{t^n}{n},$$

$$(12) \quad I_t((1-t)^n - 1) = \sum_{j=1}^n \frac{(1-t)^j - 1}{j},$$

$$(13) \quad J_{t,s}^{\star}(t^n) = \sum_{j=1}^n \frac{t^{n-j} s^j}{j},$$

$$(14) \quad J_{t,s}^{\star}((1-t)^n - 1) = \sum_{j=1}^n \frac{(1-t)^{n-j} \{(1-s)^j - 1\}}{j}.$$

Proof. The equality (11) can be easily checked. We show the equality (12). Set $T := 1-t$. Then the left hand side of (12) equals to

$$-\int \frac{T^k - 1}{1-T} dT = \int \sum_{j=0}^{k-1} T^j dT = \sum_{j=1}^k \frac{T^j - 1}{j}.$$

By the definition of T , we obtain the equality (12). Let us show the equality (13). By the equality (10), the following equalities hold:

$$J_{t,s}^*(t^n) = \text{pr}_{t,R[[s]]}(t^n I_{t,s;R}(1)) = \text{pr}_{t,R[[s]]} \left(\sum_{j=1}^{\infty} \frac{s^j t^{n-j}}{j} \right) = \sum_{j=1}^n \frac{s^j t^{n-j}}{j}.$$

Finally, we show the equality (14). Note that the following equalities hold:

$$\begin{aligned} \frac{(1-t)^n}{t-s} &= -\frac{(1-t)^{n-1}}{1-\frac{s-1}{t-1}} = -(1-t)^{n-1} \left(\frac{1-\left(\frac{s-1}{t-1}\right)^n}{1-\frac{s-1}{t-1}} + \frac{\left(\frac{s-1}{t-1}\right)^n}{1-\frac{s-1}{t-1}} \right) \\ &= -(1-t)^{n-1} \sum_{j=0}^{n-1} \left(\frac{s-1}{t-1} \right)^j + \frac{(1-s)^n}{t-s} \\ &= -\sum_{j=0}^{n-1} (1-s)^j (1-t)^{n-j-1} + \frac{(1-s)^n}{t-s}. \end{aligned}$$

As $J_{t,s}^*(f(s)) = \text{pr}_{t,R[[s]]}(\int \frac{f(s)ds}{t-s}) = 0$ for each $f(s) \in R[s]$ by the equality (10), we have

$$J_{t,s}^*((1-t)^n - 1) = J_{t,s}^*((1-t)^n) = \int \left(-\sum_{j=0}^{n-1} (1-s)^j (1-t)^{n-j-1} \right) ds = \sum_{j=1}^n \frac{(1-t)^{n-j} \{(1-s)^j - 1\}}{j}.$$

This completes the proof of the lemma. \square

Before we give the lemma for $J_{t,s}^N$, we prepare the following two auxiliary lemmas:

Lemma 2.16 (cf. [24, Proof of Lemma 4.1]). *Let N be a positive integer. We have the following polynomial identities in $R[t]$:*

$$(15) \quad \sum_{n=1}^N (-1)^n \binom{N}{n} \frac{t^n}{n} = \sum_{n=1}^N \frac{(1-t)^n - 1}{n},$$

$$(16) \quad \sum_{n=1}^N \frac{t^n}{n} = \sum_{n=1}^N (-1)^n \binom{N}{n} \frac{(1-t)^n - 1}{n}.$$

Proof. First, we remark that $(1-t)^N - 1 = \sum_{n=1}^N \binom{N}{n} (-t)^n$. Then by applying I_t to both sides and using Lemma 2.15 (11) and (12), we obtain the identity (15). The identity (16) is obtained by the substitution $t \mapsto 1-t$ and the Euler's identity (3), which is a special case of the identity (15). \square

Lemma 2.17. *Let j and n be non-negative integers satisfying $j \leq n$. Then we have the following polynomial identity in $R[t]$:*

$$\sum_{k=j}^n \binom{n}{k} \binom{k}{j} t^k = \binom{n}{j} t^j (1+t)^{n-j}.$$

Proof. By the binomial expansion, we have

$$(t + s + ts)^n = \{t + (1 + t)s\}^n = \sum_{j=0}^n \binom{n}{j} t^j (1 + t)^{n-j} s^{n-j}.$$

On the other hand, we have

$$\begin{aligned} (t + s + ts)^n &= \{t(1 + s) + s\}^n = \sum_{k=0}^n \binom{n}{k} t^k (1 + s)^k s^{n-k} \\ &= \sum_{k=0}^n \binom{n}{k} t^k \sum_{j=0}^k \binom{k}{j} s^{k-j} s^{n-k} = \sum_{j=0}^n \left\{ \sum_{k=j}^n \binom{n}{k} \binom{k}{j} t^k \right\} s^{n-j} \end{aligned}$$

Compare the coefficient of s^{n-j} . □

Lemma 2.18. *Let N and n be positive integers. Then we have the following identities in $R[t^{\pm 1}, s]$:*

$$(17) \quad J_{t,s}^N(t^n) = \sum_{j=n+1}^N \frac{s^j t^{n-j}}{j},$$

$$(18) \quad J_{t,s}^N((1-t)^n - 1) = - \sum_{j=1}^n \frac{(1-t)^{n-j} \{(1-s)^j - 1\}}{j} + \left(\sum_{j=1}^N \frac{(s/t)^j}{j} \right) \{(1-t)^n - 1\}.$$

Here, we understand the summation in the right hand side of the equality (17) as 0 if $n+1$ is greater than N .

Proof. The equality (17) is an immediate consequence of the equality (10). We show the equality (18). By the equality

$$I_{t,s;R}((1-t)^n) = (1-t)^n \sum_{j=1}^{\infty} \frac{s^j t^{-j}}{j} = \sum_{k=0}^n (-1)^k \binom{n}{k} \left(\sum_{j=1}^{\infty} \frac{s^j t^{k-j}}{j} \right),$$

we have

$$\begin{aligned} (19) \quad J_{t,s}^N((1-t)^n) &= \sum_{k=0}^n (-1)^k \binom{n}{k} \left(\sum_{j=k+1}^N \frac{s^j t^{k-j}}{j} \right) = \sum_{k=0}^n (-1)^k \binom{n}{k} \left(\sum_{j=1}^N \frac{s^j t^{k-j}}{j} - \sum_{j=1}^k \frac{s^j t^{k-j}}{j} \right) \\ &= \sum_{j=1}^N \frac{(s/t)^j}{j} (1-t)^n - \sum_{n \geq k \geq j \geq 1} (-1)^k \binom{n}{k} \frac{s^j t^{k-j}}{j}. \end{aligned}$$

Since the equality

$$\sum_{j=1}^k \frac{(s/t)^j}{j} = \sum_{j=1}^k (-1)^j \binom{k}{j} \frac{(1-s/t)^j - 1}{j}$$

holds by Lemma 2.16 (16), we have

$$\sum_{n \geq k \geq j \geq 1} (-1)^k \binom{n}{k} \frac{s^j t^{k-j}}{j} = \sum_{n \geq k \geq j \geq 1} (-1)^{j+k} \binom{n}{k} \binom{k}{j} \frac{(1-s/t)^j - 1}{j} t^k.$$

Furthermore, by Lemma 2.17, we have

$$\begin{aligned}
\sum_{n \geq k \geq j \geq 1} (-1)^{j+k} \binom{n}{k} \binom{k}{j} \frac{(1-s/t)^j - 1}{j} t^k &= \sum_{j=1}^n \binom{n}{j} t^j (1-t)^{n-j} \frac{(1-s/t)^j - 1}{j} \\
&= \sum_{j=1}^n \binom{n}{j} (1-t)^{n-j} \frac{(t-s)^j - t^j}{j} \\
&= (1-t)^n \sum_{j=1}^n \binom{n}{j} \frac{1}{j} \left\{ \left(\frac{t-s}{1-t} \right)^j - \left(\frac{t}{1-t} \right)^j \right\}.
\end{aligned}$$

Therefore, according to Lemma 2.16 (15), we can delete the binomial coefficients completely as follows:

$$\begin{aligned}
\sum_{n \geq k \geq j \geq 1} (-1)^{j+k} \binom{n}{k} \binom{k}{j} \frac{(1-s/t)^j - 1}{j} t^k &= (1-t)^n \sum_{j=1}^n \frac{1}{j} \left\{ \left(1 - \frac{s-t}{1-t} \right)^j - \left(1 - \frac{-t}{1-t} \right)^j \right\} \\
&= (1-t)^n \sum_{j=1}^n \frac{1}{j} \left\{ \left(\frac{1-s}{1-t} \right)^j - \left(\frac{1}{1-t} \right)^j \right\} \\
&= \sum_{j=1}^n \frac{(1-t)^{n-j} \{ (1-s)^j - 1 \}}{j}.
\end{aligned}$$

Hence, we have the desired identity by the equalities (19) and $J_{t,s}^N(1) = \sum_{j=1}^N \frac{(s/t)^j}{j}$. \square

2.4. Proofs of Theorem 2.5 and Theorem 2.10.

Proof of Theorem 2.5. We show this theorem by the induction on the weight w of the index. If $w = 1$, the assertion of the theorem is nothing but Lemma 2.16. We show only the equality (4) because the proof of the equality (5) is completely the same. Now, we assume that the assertion holds for an index $\mathbb{k} = (k_1, \dots, k_m)$. Then it is sufficient to show that the assertions also hold for the indices $\mathbb{k}_1 := (k_1, \dots, k_m+1)$ and $\mathbb{k}_2 := (k_1, \dots, k_m, 1)$.

First, we consider the case \mathbb{k}_1 . By Lemma 2.15 (11), we have

$$I_{t_m} \left(\sum' (-1)^{n_1} \binom{N}{n_1} \frac{t_1^{n_1-n_2} \dots t_{m-1}^{n_{m-1}-n_m} t_m^{n_m}}{n_1^{k_1} \dots n_m^{k_m}} \right) = \sum' (-1)^{n_1} \binom{N}{n_1} \frac{t_1^{n_1-n_2} \dots t_{m-1}^{n_{m-1}-n_m} t_m^{n_m}}{n_1^{k_1} \dots n_m^{k_m+1}},$$

where the summation \sum' runs over $N \geq n_1 \geq \dots \geq n_m \geq 1$ and $I_{t_m} := I_{t_m; R[t_1, \dots, t_{m-1}]}$. On the other hand, by Lemma 2.15 (12), we have

$$\begin{aligned}
&I_{t_m} \left(\sum_{N \geq n_1 \geq \dots \geq n_w \geq 1} \frac{(1-t_1)^{n_{l_1}-n_{l_1+1}} \dots (1-t_{m-1})^{n_{l_{m-1}}-n_{l_{m-1}+1}} \{(1-t_m)^{n_{l_m}} - 1\}}{n_1 \dots n_w} \right) \\
&= \sum_{N \geq n_1 \geq \dots \geq n_{w+1} \geq 1} \frac{(1-t_1)^{n_{l_1}-n_{l_1+1}} \dots (1-t_{m-1})^{n_{l_{m-1}}-n_{l_{m-1}+1}} \{(1-t_m)^{n_{l_m+1}} - 1\}}{n_1 \dots n_{w+1}},
\end{aligned}$$

where $l_1 = k_1, l_2 = k_1 + k_2, \dots, l_m = k_1 + \dots + k_m (= w)$. Thus, the equality (4) in the theorem also holds for \mathbb{k}_1 by the induction hypothesis.

Next, we check the equality for the index \mathbb{k}_2 . By Lemma 2.15 (13), we have

$$J_{t_m, t_{m+1}}^* \left(\sum' (-1)^{n_1} \binom{N}{n_1} \frac{t_1^{n_1-n_2} \dots t_{m-1}^{n_{m-1}-n_m} t_m^{n_m}}{n_1^{k_1} \dots n_m^{k_m}} \right) = \sum'' (-1)^{n_1} \binom{N}{n_1} \frac{t_1^{n_1-n_2} \dots t_m^{n_m-n_{m+1}} t_{m+1}^{n_{m+1}}}{n_1^{k_1} \dots n_m^{k_m} n_{m+1}},$$

where the summation \sum'' runs over $N \geq n_1 \geq \dots \geq n_{m+1} \geq 1$ and $J_{t_m, t_{m+1}}^* := J_{t_m, t_{m+1}; R[t_1, \dots, t_{m-1}]}$. On the other hand, by Lemma 2.15 (14), we have

$$\begin{aligned} & J_{t_m, t_{m+1}}^* \left(\sum_{N \geq n_1 \geq \dots \geq n_w \geq 1} \frac{(1-t_1)^{n_{l_1}-n_{l_1+1}} \dots (1-t_{m-1})^{n_{l_{m-1}}-n_{l_{m-1}+1}} \{(1-t_m)^{n_{l_m}} - 1\}}{n_1 \dots n_w} \right) \\ &= \sum_{N \geq n_1 \geq \dots \geq n_{w+1} \geq 1} \frac{(1-t_1)^{n_{l_1}-n_{l_1+1}} \dots (1-t_m)^{n_{l_m}-n_{l_m+1}} \{(1-t_{m+1})^{n_{l_{m+1}}} - 1\}}{n_1 \dots n_{w+1}}. \end{aligned}$$

Using the induction hypothesis, the assertion of the equality (4) holds for the index \mathbb{k}_2 . This completes the proof of Theorem 2.5. \square

Proof of Theorem 2.10. We show this theorem by the induction on the weight w of the index. If $w = 1$, the assertion of the theorem is nothing but Lemma 2.16. We show only the equality (7) because the proof of the equality (8) is completely the same. Now, we assume that the assertion holds for an index $\mathbb{k} = (k_1, \dots, k_m)$. Then it is sufficient to show that the assertions also hold for the indices $\mathbb{k}_1 := (k_1+1, \dots, k_m)$ and $\mathbb{k}_2 := (1, k_1, \dots, k_m)$.

First, we consider the case \mathbb{k}_1 . By Lemma 2.15 (11), we have

$$I_{t_1} \left(\sum' (-1)^{n_m} \binom{N}{n_m} \frac{(t_1/t_2)^{n_1} \dots (t_{m-1}/t_m)^{n_{m-1}} t_m^{n_m}}{n_1^{k_1} \dots n_m^{k_m}} \right) = \sum' (-1)^{n_m} \binom{N}{n_m} \frac{(t_1/t_2)^{n_1} \dots (t_{m-1}/t_m)^{n_{m-1}} t_m^{n_m}}{n_1^{k_1+1} \dots n_m^{k_m}},$$

where the summation \sum' runs over $N+1 > n_1 > \dots > n_m > 0$ and $I_{t_1} := I_{t_1; R[t_2^{\pm 1}, \dots, t_m^{\pm 1}]}$. On the other hand, by Lemma 2.15 (12), we have

$$\begin{aligned} & I_{t_1}(\text{R. H. S. of the equality (7)}) = \\ & (-1)^{m-1} \sum_{N \geq n_1 \geq \dots \geq n_{w+1} \geq 1} \frac{(1-t_m)^{n_{l_1}-n_{l_1+1}} \dots (1-t_2)^{n_{l_{m-1}}-n_{l_{m-1}+1}} \{(1-t_1)^{n_{l_m}} - 1\}}{n_1 \dots n_{w+1}} \\ & + \sum_{j=1}^{m-1} (-1)^{m-j-1} \left(\sum_{N+1 > n_1 > \dots > n_j > 0} \frac{(t_1/t_2)^{n_1} \dots (t_j/t_{j+1})^{n_j}}{n_1^{k_1+1} \dots n_j^{k_j}} \right) \times \\ & \left(\sum_{N \geq n_1 \geq \dots \geq n_{l_{m-j}} \geq 1} \frac{(1-t_m)^{n_{l_1}-n_{l_1+1}} \dots (1-t_{j+2})^{n_{l_{m-j-1}}-n_{l_{m-j-1}+1}} \{(1-t_{j+1})^{n_{l_{m-j}}} - 1\}}{n_1 \dots n_{l_{m-j}}} \right) \end{aligned}$$

where $l_1 = k_m, l_2 = k_m + k_{m-1}, \dots, l_m = k_m + \dots + k_1 (= w)$. Thus, the equality (7) in the theorem also holds for the index \mathbb{k}_1 by the induction hypothesis.

Next, we check the equality for the index \mathbb{k}_2 . By Lemma 2.18 (17), we have

$$J_{t_1, t_0}^N \left(\sum' (-1)^{n_m} \binom{N}{n_m} \frac{(t_1/t_2)^{n_1} \dots (t_{m-1}/t_m)^{n_{m-1}} t_m^{n_m}}{n_1^{k_1} \dots n_m^{k_m}} \right) = \sum'' (-1)^{n_m} \binom{N}{n_m} \frac{(t_0/t_1)^{n_0} \dots (t_{m-1}/t_m)^{n_{m-1}} t_m^{n_m}}{n_0 n_1^{k_1} \dots n_m^{k_m}},$$

where the summation \sum'' runs over $N+1 > n_0 > n_1 > \cdots > n_m > 0$ and $J_{t_1, t_0}^N := J_{t_1, t_0; R[t_2^{\pm 1}, \dots, t_m^{\pm 1}]}^N$. On the other hand, by Lemma 2.18 (18), we have

$$\begin{aligned}
& (-1)^m J_{t_1, t_0}^N (\text{R. H. S. of the equality (7)}) = \\
& \sum_{N \geq n_1 \geq \cdots \geq n_{w+1} \geq 1} \frac{(1-t_m)^{n_{l_1}-n_{l_1+1}} \cdots (1-t_1)^{n_{l_m}-n_{l_m+1}} \{(1-t_0)^{n_{l_m+1}} - 1\}}{n_1 \cdots n_{w+1}} \\
& - \left(\sum_{n_0=1}^N \frac{(t_0/t_1)^{n_0}}{n_0} \right) \left(\sum_{N \geq n_1 \geq \cdots \geq n_w \geq 1} \frac{(1-t_m)^{n_{l_1}-n_{l_1+1}} \cdots (1-t_2)^{n_{l_{m-1}}-n_{l_{m-1}+1}} \{(1-t_1)^{n_{l_m}} - 1\}}{n_1 \cdots n_w} \right) \\
& + \sum_{j=1}^{m-1} (-1)^{j-1} \left(\sum_{N+1 > n_0 > n_1 > \cdots > n_j > 0} \frac{(t_0/t_1)^{n_0} \cdots (t_j/t_{j+1})^{n_j}}{n_0 n_1^{k_1} \cdots n_j^{k_j}} \right) \times \\
& \left(\sum_{N \geq n_1 \geq \cdots \geq n_{l_m-j} \geq 1} \frac{(1-t_m)^{n_{l_1}-n_{l_1+1}} \cdots (1-t_{j+2})^{n_{l_{m-j-1}}-n_{l_{m-j-1}+1}} \{(1-t_{j+1})^{n_{l_m-j}} - 1\}}{n_1 \cdots n_{l_m-j}} \right)
\end{aligned}$$

Using the induction hypothesis, the assertion of the equality (7) holds for the index \mathbb{k}_2 . This completes the proof of Theorem 2.10. \square

3. FUNCTIONAL EQUATIONS OF FINITE MULTIPLE POLYLOGARITHMS

3.1. The ring $\mathcal{A}_{n,R}^\Sigma$. Kaneko and Zagier defined finite multiple zeta(-star) values as elements of the \mathbb{Q} -algebra $\mathcal{A} = (\prod_p \mathbb{F}_p) \otimes_{\mathbb{Z}} \mathbb{Q}$ where p runs over the set of all rational primes. See [10] and their forthcoming paper [11]. The ring \mathcal{A} has been used in a different context by Kontsevich [15, 2.2]. Now, we introduce more general rings of such a type for the definition of finite multiple polylogarithms.

Definition 3.1. Let R be a commutative ring and Σ a family of ideals of R . Then we define the ring $\mathcal{A}_{n,R}^\Sigma$ for each positive integer n as follows:

$$\mathcal{A}_{n,R}^\Sigma := \left(\prod_{I \in \Sigma} R/I^n \right) / \left(\bigoplus_{I \in \Sigma} R/I^n \right).$$

Since we use only the case $\Sigma = \{pR \mid p \text{ is a rational prime}\}$, we omit the notation Σ in the rest of this paper. We denote $\mathcal{A}_{n,\mathbb{Z}} = \mathcal{A}_n$ and $\mathcal{A}_{1,R} = \mathcal{A}_R$. Then the ring $\mathcal{A}_{1,\mathbb{Z}}$ coincides with \mathcal{A} . We will define the finite multiple polylogarithms as elements of $\mathcal{A}_{n,\mathbb{Z}[t_1, \dots, t_m]}$ (that is not equal to the polynomial ring $\mathcal{A}_n[t_1, \dots, t_m]$). Note that $\mathcal{A}_{n,R}$ is a \mathbb{Q} -algebra in some important case even if R is not a \mathbb{Q} -algebra.

Example 3.2. We often use

$$\mathcal{A}_{\mathbb{Z}[t]} = \left(\prod_p \mathbb{F}_p[t] \right) / \left(\bigoplus_p \mathbb{F}_p[t] \right)$$

in the next section. Here, the direct product and the direct sum run over the all rational primes and $\mathcal{A}_{\mathbb{Z}[t]}$ coincides with \mathcal{B} defined by Ono and Yamamoto in their paper [21].

We denote each element of $\mathcal{A}_{n,R}$ as $(a_p)_p$ where $a_p \in R/p^n R$, so $(a_p)_p = (b_p)_p$ holds if and only if $a_p = b_p$ for all but finitely many rational primes p . We may use the notation $(a_p)_p \in \mathcal{A}_{n,R}$ even if a_p is not defined for finitely many rational primes p . See Example

3.3 (3). We define the element $\mathbf{p}_n \in \mathcal{A}_{n,R}$ to be $(p \bmod p^n)_p$ and we denote it by \mathbf{p} when n is clear from the context. Note that $\mathcal{A}_n/\mathbf{p}^m \mathcal{A}_n = \mathcal{A}_m$ for $n \geq m$.

Example 3.3. We give some typical examples of elements of $\mathcal{A}_{n,R}$.

- (1) $t^{\mathbf{p}} := (t^p)_p \in \mathcal{A}_{n,\mathbb{Z}[t]}$.
- (2) For any rational number r , we define $r^{\mathbf{p}}$ (resp. r) to be $(r^p)_p$ (resp. $(r)_p$) in \mathcal{A} . Then $r^{\mathbf{p}} = r \in \mathcal{A}$ holds by Fermat's little theorem.
- (3) Let k be a positive integer greater than 2. We use the notation $B_{\mathbf{p}-k}$ as

$$B_{\mathbf{p}-k} = (B_{p-k} \bmod p^n)_p \in \mathcal{A}_n,$$

where B_m is the m -th Bernoulli number defined by

$$\frac{te^t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}.$$

By the von Staudt-Clausen theorem [29, Theorem 5.10], B_{p-k} is a p -adic integer for a rational prime greater than k . Therefore, the element $B_{\mathbf{p}-k}$ is well-defined as an element of \mathcal{A}_n and conjecturally, it is non-zero by the conjecture that there are infinitely many regular primes. For later use, we define \widehat{B}_m to be $\frac{B_m}{m}$ for a positive integer m . For example, we use the notation $\widehat{B}_{\mathbf{p}-k}$ as $(\frac{B_{p-k}}{p-k} \bmod p^n)_p \in \mathcal{A}_n$.

The following lemma will be used for deducing the functional equations of the finite multiple polylogarithms from the generalizations of Euler's identity obtained in Section 2.

Lemma 3.4. *Let n be a positive integer. Then we have*

$$(20) \quad (-1)^n \binom{\mathbf{p}-1}{n} = 1 - \mathbf{p}H_n \text{ in } \mathcal{A}_2,$$

where $H_n = \sum_{j=1}^n 1/j$ is the n -th harmonic number.

Let p be an odd prime number greater than n . Then the following congruence is satisfied:

$$(21) \quad H_{p-n-1} \equiv H_n \pmod{p}.$$

Here, we define H_0 to be 1.

Proof. For each prime number $p > n$, we have the following congruence:

$$(-1)^n \binom{p-1}{n} = \prod_{j=1}^n \left(1 - \frac{p}{j}\right) \equiv 1 - pH_n \pmod{p^2}.$$

Therefore the first assertion holds. Let $p > n+1$. By the substitution $n \mapsto p-n$, we have

$$(22) \quad H_{p-n-1} = \sum_{k=1}^{p-n-1} \frac{1}{k} = \sum_{k=n+1}^{p-1} \frac{1}{p-k} \equiv - \sum_{k=n+1}^{p-1} \frac{1}{k} = -(H_{p-1} - H_n) \equiv H_n \pmod{p}.$$

□

3.2. Definitions and functional equations of finite multiple polylogarithms. First, we recall the definition of the finite multiple zeta(-star) values (FMZ(S)Vs).

Definition 3.5. Let n be a positive integer and $\mathbb{k} = (k_1, \dots, k_m)$ an index. Then we define the finite multiple zeta value $\zeta_{\mathcal{A}_n}(\mathbb{k})$ by

$$\zeta_{\mathcal{A}_n}(\mathbb{k}) := \left(\sum_{p > n_1 > \dots > n_m > 0} \frac{1}{n_1^{k_1} \dots n_m^{k_m}} \bmod p^n \right)_p \in \mathcal{A}_n$$

and we define the finite multiple zeta-star value $\zeta_{\mathcal{A}_n}^*(\mathbb{k})$ by

$$\zeta_{\mathcal{A}_n}^*(\mathbb{k}) := \left(\sum_{p-1 \geq n_1 \geq \dots \geq n_m \geq 1} \frac{1}{n_1^{k_1} \dots n_m^{k_m}} \bmod p^n \right)_p \in \mathcal{A}_n.$$

Remark 3.6. Several people use the notation $\zeta_{\mathcal{A}_n}^\bullet(\mathbb{k})$ instead of $\zeta_{\mathcal{A}_n}^\bullet(\bar{\mathbb{k}})$ for $\bullet \in \{\emptyset, \star\}$. Therefore we have to be careful when we read other papers.

For later use, we summarize known results about FMZ(S)Vs.

Proposition 3.7. Let m, k, k_1, k_2, k_3 be positive integers, $\mathbb{k} = (k_1, \dots, k_m)$ an index and $\bullet \in \{\emptyset, \star\}$. Then the following equalities hold:

$$(23) \quad \zeta_{\mathcal{A}_2}(\{k\}^m) = (-1)^{m-1} \zeta_{\mathcal{A}_2}^*(\{k\}^m) = (-1)^{m-1} k \frac{B_{\mathbf{p}-mk-1}}{mk+1} \mathbf{p},$$

$$(24) \quad \zeta_{\mathcal{A}_3}(k) = \begin{cases} \binom{k+1}{2} \widehat{B}_{\mathbf{p}-k-2} \mathbf{p}^2 & \text{if } k \text{ is odd,} \\ k \left(\widehat{B}_{2\mathbf{p}-k-2} - 2\widehat{B}_{\mathbf{p}-k-1} \right) \mathbf{p} & \text{if } k \text{ is even,} \end{cases}$$

$$(25) \quad \zeta_{\mathcal{A}_4}(k) = \begin{cases} -\binom{k+1}{2} \left(\widehat{B}_{2\mathbf{p}-k-3} - 2\widehat{B}_{\mathbf{p}-k-2} \right) \mathbf{p}^2 & \text{if } k \text{ is odd,} \\ -k \left(\widehat{B}_{3\mathbf{p}-k-3} - 3\widehat{B}_{2\mathbf{p}-k-2} + 3\widehat{B}_{\mathbf{p}-k-1} \right) \mathbf{p} - \binom{k+2}{3} \widehat{B}_{\mathbf{p}-k-3} \mathbf{p}^3 & \text{if } k \text{ is even,} \end{cases}$$

$$(26) \quad \zeta_{\mathcal{A}}^\bullet(k_1, k_2) = (-1)^{k_1} \binom{k_1 + k_2}{k_1} \frac{B_{\mathbf{p}-k_1-k_2}}{k_1 + k_2},$$

$$(27) \quad \zeta_{\mathcal{A}_2}(k_1, k_2) = \frac{1}{2} \left\{ (-1)^{k_2} k_1 \binom{w+1}{k_2} - (-1)^{k_1} k_2 \binom{w+1}{k_1} - w \right\} \frac{B_{\mathbf{p}-w-1}}{w+1} \mathbf{p}$$

if $w = k_1 + k_2$ is even,

$$(28) \quad \zeta_{\mathcal{A}_2}^*(k_1, k_2) = \frac{1}{2} \left\{ (-1)^{k_2} k_1 \binom{w+1}{k_2} - (-1)^{k_1} k_2 \binom{w+1}{k_1} + w \right\} \frac{B_{\mathbf{p}-w-1}}{w+1} \mathbf{p}$$

if $w = k_1 + k_2$ is even,

$$(29) \quad \zeta_{\mathcal{A}}(k_1, k_2, k_3) = -\zeta_{\mathcal{A}}^*(k_1, k_2, k_3) = \frac{1}{2} \left\{ (-1)^{k_3} \binom{w'}{k_3} - (-1)^{k_1} \binom{w'}{k_1} \right\} \frac{B_{\mathbf{p}-w'}}{w'}$$

if $w' = k_1 + k_2 + k_3$ is odd,

$$(30) \quad \sum_{\sigma \in S_m} \zeta_{\mathcal{A}}^\bullet(\sigma(\mathbb{k})) = 0,$$

where S_m denotes the m -th symmetric group and $\sigma(\mathbb{k}) := (k_{\sigma(1)}, \dots, k_{\sigma(m)})$,

$$(31) \quad \zeta_{\mathcal{A}}^{\bullet}(\mathbb{k}) = (-1)^{\text{wt}(\mathbb{k})} \zeta_{\mathcal{A}}^{\bullet}(\overline{\mathbb{k}}),$$

$$(32) \quad \zeta_{\mathcal{A}}^{\star}(\mathbb{k}) = -\zeta_{\mathcal{A}}^{\star}(\mathbb{k}^{\vee}),$$

$$(33) \quad \zeta_{\mathcal{A}}^{\star}(k_1, \{1\}^{k_2-1}) = (-1)^{k_1} \zeta_{\mathcal{A}}(k_1, \{1\}^{k_2-1}).$$

Proof. The equalities (23), (26), (27), (28), and (29) are [31, Theorem 1.6], [31, Theorem 3.1] (cf. [8, Theorem 6.1]), [31, Theorem 3.2], [31, Theorem 3.2], and [31, Theorem 3.5] (cf. [8, Theorem 6.2]), respectively. The equalities (24) and (25) are [23, Theorem 5.1] and [23, Remark 5.1], respectively. The equalities (30), (31), (32), and (33) are [8, Theorem 4.4], [8, Theorem 4.5], [8, Theorem 4.6], and [8, Theorem 5.1], respectively. \square

Now, we define the finite multiple polylogarithms which are main objects in this paper.

Definition 3.8. Let n be a positive integer and $\mathbb{k} = (k_1, \dots, k_m)$ an index. Then we define the various multi-variable finite multiple polylogarithms as follows:

The finite harmonic multiple polylogarithm (FHMP):

$$\mathcal{L}_{\mathcal{A}_n, \mathbb{k}}^*(t_1, \dots, t_m) := \left(\sum_{p > n_1 > \dots > n_m > 0} \frac{t_1^{n_1} \dots t_m^{n_m}}{n_1^{k_1} \dots n_m^{k_m}} \bmod p^n \right)_p \in \mathcal{A}_{n, \mathbb{Z}[t_1, \dots, t_m]},$$

The finite harmonic star-multiple polylogarithm (FHSMP):

$$\mathcal{L}_{\mathcal{A}_n, \mathbb{k}}^{*, \star}(t_1, \dots, t_m) := \left(\sum_{p-1 \geq n_1 \geq \dots \geq n_m \geq 1} \frac{t_1^{n_1} \dots t_m^{n_m}}{n_1^{k_1} \dots n_m^{k_m}} \bmod p^n \right)_p \in \mathcal{A}_{n, \mathbb{Z}[t_1, \dots, t_m]},$$

The finite shuffle multiple polylogarithm (FSMP):

$$\mathcal{L}_{\mathcal{A}_n, \mathbb{k}}^{\text{III}}(t_1, \dots, t_m) := \left(\sum_{p > n_1 > \dots > n_m > 0} \frac{t_1^{n_1-n_2} \dots t_{m-1}^{n_{m-1}-n_m} t_m^{n_m}}{n_1^{k_1} \dots n_m^{k_m}} \bmod p^n \right)_p \in \mathcal{A}_{n, \mathbb{Z}[t_1, \dots, t_m]},$$

The finite shuffle star-multiple polylogarithm (FSSMP):

$$\mathcal{L}_{\mathcal{A}_n, \mathbb{k}}^{\text{III}, \star}(t_1, \dots, t_m) := \left(\sum_{p-1 \geq n_1 \geq \dots \geq n_m \geq 1} \frac{t_1^{n_1-n_2} \dots t_{m-1}^{n_{m-1}-n_m} t_m^{n_m}}{n_1^{k_1} \dots n_m^{k_m}} \bmod p^n \right)_p \in \mathcal{A}_{n, \mathbb{Z}[t_1, \dots, t_m]}.$$

We also define 1-variable finite (star-)multiple polylogarithms (F(S)MP) as follows:

$$\begin{aligned} \mathcal{L}_{\mathcal{A}_n, \mathbb{k}}^{\bullet}(t) &:= \mathcal{L}_{\mathcal{A}_n, \mathbb{k}}^{*, \bullet}(t, \{1\}^{m-1}) = \mathcal{L}_{\mathcal{A}_n, \mathbb{k}}^{\text{III}, \bullet}(\{t\}^m) \in \mathcal{A}_{n, \mathbb{Z}[t]}, \\ \widetilde{\mathcal{L}}_{\mathcal{A}_n, \mathbb{k}}^{\bullet}(t) &:= \mathcal{L}_{\mathcal{A}_n, \mathbb{k}}^{*, \bullet}(\{1\}^{m-1}, t) = \mathcal{L}_{\mathcal{A}_n, \mathbb{k}}^{\text{III}, \bullet}(\{1\}^{m-1}, t) \in \mathcal{A}_{n, \mathbb{Z}[t]}, \end{aligned}$$

where $\bullet \in \{\emptyset, \star\}$. We call $\mathcal{L}_{\mathcal{A}_n, m}^{\bullet}(t) = \widetilde{\mathcal{L}}_{\mathcal{A}_n, m}^{\bullet}(t)$ the m -th finite polylogarithm.

Remark 3.9. The original definition of the m -th finite polylogarithm by Elbaz-Vincent and Gangl is the p -component of $\mathcal{L}_{\mathcal{A}, m}(t)$ in $\mathbb{F}_p[t]$ (cf. [4, Definition 5.1]).

Remark 3.10. Let R be a commutative ring. For any subset $\{i_1, \dots, i_h\}$ of $\{1, \dots, m\}$ and $r_1, \dots, r_h \in R$, the substitution mapping

$$\mathcal{A}_{n, \mathbb{Z}[t_1, \dots, t_m]} \longrightarrow \mathcal{A}_{n, R[t_{j_1}, \dots, t_{j_h}]}$$

defined by

$$(f_p(t_1, \dots, t_m))_p \mapsto (f_p(t_1, \dots, t_m)|_{t_{i_1}=r_1, \dots, t_{i_h}=r_h})_p$$

where $\{j_1, \dots, j_h\}$ is the complement of $\{i_1, \dots, i_h\}$ with respect to $\{1, \dots, m\}$. For example, we have

$$\mathcal{L}_{\mathcal{A}_n, \mathbb{k}}^\bullet(1) = \tilde{\mathcal{L}}_{\mathcal{A}_n, \mathbb{k}}^\bullet(1) = \zeta_{\mathcal{A}_n}^\bullet(\mathbb{k}) \in \mathcal{A}_n$$

for $\bullet \in \{\emptyset, \star\}$. Our definition of FMPs is natural in this sense.

The following proposition is a generalization of Proposition 3.7 (31) (cf. [28, Lemma 5.4]):

Proposition 3.11 (Reversal formulas). *Let $\mathbb{k} = (k_1, \dots, k_m)$ be an index and $\bullet \in \{\emptyset, \star\}$. Then we have the following equality in $\mathcal{A}_{2, \mathbb{Z}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]}$:*

$$(34) \quad \mathcal{L}_{\mathcal{A}_2, \mathbb{k}}^{\bullet}(t_1, \dots, t_m) = (-1)^{\text{wt}(\mathbb{k})} (t_1 \cdots t_m)^{\mathbf{p}} \left(\mathcal{L}_{\mathcal{A}_2, \bar{\mathbb{k}}}^{\bullet}(t_m^{-1}, \dots, t_1^{-1}) + \mathbf{p} \sum_{i=1}^m k_i \mathcal{L}_{\mathcal{A}_2, \mathbb{k} \oplus \mathbf{e}_i}^{\bullet}(t_m^{-1}, \dots, t_1^{-1}) \right).$$

In particular, we have

$$(35) \quad \mathcal{L}_{\mathcal{A}, \mathbb{k}}^{\bullet}(t_1, \dots, t_m) = (-1)^{\text{wt}(\mathbb{k})} (t_1 \cdots t_m)^{\mathbf{p}} \mathcal{L}_{\mathcal{A}, \bar{\mathbb{k}}}^{\bullet}(t_m^{-1}, \dots, t_1^{-1}),$$

$$(36) \quad \mathcal{L}_{\mathcal{A}, \mathbb{k}}^\bullet(t) = (-1)^{\text{wt}(\mathbb{k})} t^{\mathbf{p}} \tilde{\mathcal{L}}_{\mathcal{A}, \bar{\mathbb{k}}}^\bullet(t^{-1}), \quad \tilde{\mathcal{L}}_{\mathcal{A}, \mathbb{k}}^\bullet(t) = (-1)^{\text{wt}(\mathbb{k})} t^{\mathbf{p}} \mathcal{L}_{\mathcal{A}, \bar{\mathbb{k}}}^\bullet(t^{-1}).$$

Proof. We show only the case $\bullet = \emptyset$ since the proof for the case $\bullet = \star$ is similar. By using the substitution trick $n_i \mapsto p - n_i$, we have

$$\begin{aligned} \mathcal{L}_{\mathcal{A}_2, \mathbb{k}}^*(t_1, \dots, t_m) &= \left(\sum_{p > n_1 > \dots > n_m > 0} \frac{t_1^{n_1} \cdots t_m^{n_m}}{n_1^{k_1} \cdots n_m^{k_m}} \bmod p^2 \right)_p \\ &= \left(\sum_{p > p-n_1 > \dots > p-n_m > 0} \frac{t_1^{p-n_1} \cdots t_m^{p-n_m}}{(p-n_1)^{k_1} \cdots (p-n_m)^{k_m}} \bmod p^2 \right)_p \\ &= (-1)^{\text{wt}(\mathbb{k})} (t_1 \cdots t_m)^{\mathbf{p}} \left(\sum_{p > n_m > \dots > n_1 > 0} \frac{(p+n_m)^{k_m} \cdots (p+n_1)^{k_1} t_m^{-n_m} \cdots t_1^{-n_1}}{n_m^{2k_m} \cdots n_1^{2k_1}} \bmod p^2 \right)_p. \end{aligned}$$

Since $(p+n_m)^{k_m} \cdots (p+n_1)^{k_1} = n_m^{k_m} \cdots n_1^{k_1} + p \sum_{i=1}^m k_i n_m^{k_m} \cdots n_i^{k_i-1} \cdots n_1^{k_1} \pmod{p^2}$, we have the equality (34). \square

Our main results in this paper are Theorem 3.12, Corollary 3.13 and Theorem 3.15 below:

Theorem 3.12. *Let $\mathbb{k} = (k_1, \dots, k_m)$ be an index of weight w . Then we have the following functional equation in $\mathcal{A}_{2, \mathbb{Z}[t_1, \dots, t_m]}$ for FSSMPs:*

$$\begin{aligned} (37) \quad & \mathcal{L}_{\mathcal{A}_2, \mathbb{k}}^{\text{III}, \star}(t_1, \dots, t_m) + \left(\mathcal{L}_{\mathcal{A}_2, \{1\} \sqcup \mathbb{k}}^{\text{III}, \star}(1, t_1, \dots, t_m) - \mathcal{L}_{\mathcal{A}_2, \mathbf{e}_1 \oplus \mathbb{k}}^{\text{III}, \star}(t_1, \dots, t_m) \right) \mathbf{p} \\ &= \mathcal{L}_{\mathcal{A}_2, \{1\}^w}^{\text{III}, \star}(\{1\}^{k_1-1}, 1-t_1, \{1\}^{k_2-1}, 1-t_2, \dots, \{1\}^{k_m-1}, 1-t_m) \\ &\quad - \mathcal{L}_{\mathcal{A}_2, \{1\}^w}^{\text{III}, \star}(\{1\}^{k_1-1}, 1-t_1, \dots, \{1\}^{k_{m-1}-1}, 1-t_{m-1}, \{1\}^{k_m}). \end{aligned}$$

Proof. By Lemma 3.4 (20), we have

$$\begin{aligned}
& \left(\sum_{p-1 \geq n_1 \geq \dots \geq n_m \geq 1} (-1)^{n_1} \binom{p-1}{n_1} \frac{t_1^{n_1-n_2} \dots t_{m-1}^{n_{m-1}-n_m} t_m^{n_m}}{n_1^{k_1} \dots n_m^{k_m}} \bmod p^2 \right)_p \\
&= \left(\sum_{p-1 \geq n_1 \geq \dots \geq n_m \geq 1} (1 - pH_{n_1}) \frac{t_1^{n_1-n_2} \dots t_{m-1}^{n_{m-1}-n_m} t_m^{n_m}}{n_1^{k_1} \dots n_m^{k_m}} \bmod p^2 \right)_p \\
&= \mathfrak{L}_{\mathcal{A}_2, \mathbb{k}}^{\text{III}, \star}(t_1, \dots, t_m) - \mathbf{p} \left(\sum_{p-1 \geq n_1 \geq \dots \geq n_m \geq 1} \frac{H_{n_1} t_1^{n_1-n_2} \dots t_{m-1}^{n_{m-1}-n_m} t_m^{n_m}}{n_1^{k_1} \dots n_m^{k_m}} \bmod p^2 \right)_p.
\end{aligned}$$

By substitutions $n_i \mapsto p - n_i$ and Lemma 3.4 (21), we have

$$\begin{aligned}
& \left(\sum_{p-1 \geq n_1 \geq \dots \geq n_m \geq 1} \frac{H_{n_1} t_1^{n_1-n_2} \dots t_{m-1}^{n_{m-1}-n_m} t_m^{n_m}}{n_1^{k_1} \dots n_m^{k_m}} \bmod p \right)_p \\
&= \left(\sum_{p-1 \geq p-n_1 \geq \dots \geq p-n_m \geq 1} \frac{H_{p-n_1} t_1^{(p-n_1)-(p-n_2)} \dots t_{m-1}^{(p-n_{m-1})-(p-n_m)} t_m^{p-n_m}}{(p-n_1)^{k_1} \dots (p-n_m)^{k_m}} \bmod p \right)_p \\
&= (-1)^{\text{wt}(\mathbb{k})} \left(\sum_{p-1 \geq n_m \geq \dots \geq n_1 \geq 1} \frac{(H_{n_1} - \frac{1}{n_1}) t_1^{n_2-n_1} \dots t_{m-1}^{n_m-n_{m-1}} t_m^{p-n_m}}{n_m^{k_m} \dots n_1^{k_1}} \bmod p \right)_p \\
&= (-1)^{\text{wt}(\mathbb{k})} \left(\sum_{p-1 \geq n_m \geq \dots \geq n_1 \geq n_0 \geq 1} \frac{t_1^{n_2-n_1} \dots t_{m-1}^{n_m-n_{m-1}} t_m^{p-n_m}}{n_m^{k_m} \dots n_1^{k_1} n_0} \right. \\
&\quad \left. - \sum_{p-1 \geq n_m \geq \dots \geq n_1 \geq 1} \frac{t_1^{n_2-n_1} \dots t_{m-1}^{n_m-n_{m-1}} t_m^{p-n_m}}{n_m^{k_m} \dots n_1^{k_1+1}} \bmod p \right)_p \\
&= (-1)^{\text{wt}(\mathbb{k})} \left(\sum_{p-1 \geq p-n_m \geq \dots \geq p-n_1 \geq p-n_0 \geq 1} \frac{t_1^{(p-n_2)-(p-n_1)} \dots t_{m-1}^{(p-n_m)-(p-n_{m-1})} t_m^{p-(p-n_m)}}{(p-n_m)^{k_m} \dots (p-n_1)^{k_1} (p-n_0)} \right. \\
&\quad \left. - \sum_{p-1 \geq p-n_m \geq \dots \geq p-n_1 \geq 1} \frac{t_1^{(p-n_2)-(p-n_1)} \dots t_{m-1}^{(p-n_m)-(p-n_{m-1})} t_m^{p-(p-n_m)}}{(p-n_m)^{k_m} \dots (p-n_1)^{k_1+1}} \bmod p \right)_p \\
&= - \left(\sum_{p-1 \geq n_0 \geq n_1 \geq \dots \geq n_m \geq 1} \frac{t_1^{n_1-n_2} \dots t_{m-1}^{n_{m-1}-n_m} t_m^{n_m}}{n_0 n_1^{k_1} \dots n_m^{k_m}} \right. \\
&\quad \left. - \sum_{p-1 \geq n_1 \geq \dots \geq n_m \geq 1} \frac{t_1^{n_1-n_2} \dots t_{m-1}^{n_{m-1}-n_m} t_m^{n_m}}{n_1^{k_1+1} \dots n_m^{k_m}} \bmod p \right)_p \\
&= - \mathfrak{L}_{\mathcal{A}, \{1\} \sqcup \mathbb{k}}^{\text{III}, \star}(1, t_1, \dots, t_m) + \mathfrak{L}_{\mathcal{A}, \mathbf{e}_1 \oplus \mathbb{k}}^{\text{III}, \star}(t_1, \dots, t_m).
\end{aligned}$$

Therefore, we have the desired functional equation by Theorem 2.5. \square

When we substitute 1 for some of the variables t_1, \dots, t_{m-1} in (37), we have the following functional equation:

Corollary 3.13. *Let $\mathbb{k} = (k_1, \dots, k_m)$ be an index, $S = \{i_1, \dots, i_h\}$ a subset of $\{1, \dots, m-1\}$, and $\{j_1, \dots, j_{h'}\}$ the complement of $\{i_1, \dots, i_h\}$ with respect to $\{1, \dots, m-1\}$. We define $\mathbb{k}_S := (k_1 + \dots + k_{i_1}, k_{i_1+1} + \dots + k_{i_2}, \dots, k_{i_h+1} + \dots + k_m)$ and $m' := \text{dep}(\mathbb{k}_S^\vee)$. Then we have the following functional equation in $\mathcal{A}_{2, \mathbb{Z}[t_{j_1}, \dots, t_{j_{h'}}]}$ for FSSMPs:*

$$(38) \quad \left\{ \mathcal{L}_{\mathcal{A}_2, \mathbb{k}}^{\text{III}, \star}(t_1, \dots, t_m) + \left(\mathcal{L}_{\mathcal{A}_2, \{1\} \sqcup \mathbb{k}}^{\text{III}, \star}(1, t_1, \dots, t_m) - \mathcal{L}_{\mathcal{A}_2, \mathbf{e}_1 \oplus \mathbb{k}}^{\text{III}, \star}(t_1, \dots, t_m) \right) \mathbf{p} \right\} \Big|_{t_{i_1} = \dots = t_{i_h} = 1} \\ = \mathcal{L}_{\mathcal{A}_2, \mathbb{k}_S^\vee}^{\text{III}, \star}(\{1\}^{l_1}, 1 - t_{j_1}, \{1\}^{l_2}, 1 - t_{j_2}, \dots, \{1\}^{l_{h'}}, 1 - t_{j_{h'}}, \{1\}^{m' - M_{h'} - 1}, 1 - t_m) \\ - \mathcal{L}_{\mathcal{A}_2, \mathbb{k}_S^\vee}^{\text{III}, \star}(\{1\}^{l_1}, 1 - t_{j_1}, \{1\}^{l_2}, 1 - t_{j_2}, \dots, \{1\}^{l_{h'}}, 1 - t_{j_{h'}}, \{1\}^{m' - M_{h'}}),$$

where $l_1 = k_1 + \dots + k_{j_1} - j_1$, $l_2 = k_{j_1+1} + \dots + k_{j_2} - j_2 + j_1$, \dots , $l_{h'} = k_{j_{h'-1}+1} + \dots + k_{j_{h'}} - j_{h'} + j_{h'-1}$, and $M_{h'} = k_1 + \dots + k_{j_{h'}} - j_{h'} + h'$.

Proof. This is obtained by combining the proof of Theorem 3.12 and Corollary 2.6. \square

Remark 3.14. In particular, we have the following functional equation in $\mathcal{A}_{2, \mathbb{Z}[t]}$ (cf. Corollary 2.7):

$$(39) \quad \tilde{\mathcal{L}}_{\mathcal{A}_2, \mathbb{k}}^{\star}(t) + (\tilde{\mathcal{L}}_{\mathcal{A}_2, \{1\} \sqcup \mathbb{k}}^{\star}(t) - \tilde{\mathcal{L}}_{\mathcal{A}_2, \mathbf{e}_1 \oplus \mathbb{k}}^{\star}(t)) \mathbf{p} = \tilde{\mathcal{L}}_{\mathcal{A}_2, \mathbb{k}^\vee}^{\star}(1 - t) - \zeta_{\mathcal{A}_2}^{\star}(\mathbb{k}^\vee).$$

Therefore, we also have the functional equation (1) in Introduction. The case $t = 1$ gives the Hoffman duality (Proposition 3.7 (32)) and its generalization in \mathcal{A}_2 ([31, Theorem 2.11]):

$$(40) \quad \zeta_{\mathcal{A}_2}^{\star}(\mathbb{k}) + (\zeta_{\mathcal{A}_2}^{\star}(\{1\} \sqcup \mathbb{k}) - \zeta_{\mathcal{A}_2}^{\star}(\mathbf{e}_1 \oplus \mathbb{k})) \mathbf{p} = -\zeta_{\mathcal{A}_2}^{\star}(\mathbb{k}^\vee).$$

Theorem 3.15. *Let n be a positive integer and $\mathbb{k} = (k_1, \dots, k_m)$ an index. Then we have the following functional equation in $\mathcal{A}_{n, \mathbb{Z}[t_1, \dots, t_m]}$:*

$$(41) \quad \sum_{j=0}^m (-1)^j \mathcal{L}_{\mathcal{A}_n, (k_1, \dots, k_j)}^{\star}(t_1, \dots, t_j) \mathcal{L}_{\mathcal{A}_n, (k_m, \dots, k_{j+1})}^{\star, \star}(t_m, \dots, t_{j+1}) = 0.$$

Here, we consider $\mathcal{L}_{\mathcal{A}_n, (k_1, \dots, k_j)}^{\star}(t_1, \dots, t_j)$ (resp. $\mathcal{L}_{\mathcal{A}_n, (k_m, \dots, k_{j+1})}^{\star, \star}(t_m, \dots, t_{j+1})$) as 1 when $j = 0$ (resp. $j = m$).

Proof. This is an immediate consequence of Theorem 2.11. \square

Corollary 3.16. *Let n , k , and m be positive integers and $\mathbb{k} = (k_1, \dots, k_m)$ an index. Then the following equalities hold:*

$$(42) \quad \mathcal{L}_{\mathcal{A}, \{k\}^m}^{\star}(\{1\}^{i-1}, t, \{1\}^{m-i}) + (-1)^m \mathcal{L}_{\mathcal{A}, \{k\}^m}^{\star, \star}(\{1\}^{m-i}, t, \{1\}^{i-1}) = 0,$$

$$(43) \quad (-1)^{m-1} \mathcal{L}_{\mathcal{A}_n, \mathbb{k}}^{\star}(t) = \tilde{\mathcal{L}}_{\mathcal{A}_n, \mathbb{k}}^{\star}(t) + \sum_{j=1}^{m-1} (-1)^j \mathcal{L}_{\mathcal{A}_n, (k_1, \dots, k_j)}^{\star}(t) \zeta_{\mathcal{A}_n}^{\star}(k_m, \dots, k_{j+1}),$$

$$(44) \quad (-1)^{m-1} \tilde{\mathcal{L}}_{\mathcal{A}_n, \mathbb{k}}^{\star}(t) = \mathcal{L}_{\mathcal{A}_n, \mathbb{k}}^{\star}(t) + \sum_{j=1}^{m-1} (-1)^j \zeta_{\mathcal{A}_n}^{\star}(k_1, \dots, k_j) \mathcal{L}_{\mathcal{A}_n, (k_m, \dots, k_{j+1})}^{\star}(t),$$

$$(45) \quad \sum_{j=0}^m (-1)^j \zeta_{\mathcal{A}_n}^{\star}(k_1, \dots, k_j) \zeta_{\mathcal{A}_n}^{\star}(k_m, \dots, k_{j+1}) = 0,$$

Here, we consider $\zeta_{\mathcal{A}_n}^{\bullet}(\emptyset)$ as 1 for $\bullet \in \{\emptyset, \star\}$.

Proof. We obtain the equality (42) by the substitution $t_1 = \cdots = t_{i-1} = t_{i+1} = \cdots = t_m = 1, t_i = t$ and Lemma 3.7 (23). The equalities (43), (44), and (45) are clear. \square

Remark 3.17. The equality (42) has been proved by Tauraso and J. Zhao ([28, Lemma 5.9]). By considering the case $\mathbb{k} = (k_1, \{1\}^{k_2-1})$ and $n = 1$ in the equality (45), we have

$$\zeta_{\mathcal{A}}(k_1, \{1\}^{k_2-1}) + (-1)^{k_2} \zeta_{\mathcal{A}}^*(\{1\}^{k_2-1}, k_1) = 0$$

since $\zeta_{\mathcal{A}}^*(\{1\}^k) = 0$ for every positive integer k . Therefore, the equality (45) is a generalization of Proposition 3.7 (33) since

$$\zeta_{\mathcal{A}}^*(\{1\}^{k_2-1}, k_1) = (-1)^{k_1+k_2-1} \zeta_{\mathcal{A}}^*(k_1, \{1\}^{k_2-1})$$

by Proposition 3.7 (31). The equality (45) and its analogue for the usual multiple zeta values (the equality (9)) are consequences of the explicit formula of the antipode of the harmonic algebra or the Hopf algebra of quasi-symmetric functions ([7, Theorem 3.2] or [8, Theorem 3.1]). See also [33, Theorem 3], [9, Proposition 6], [12, Proposition 7,1], and [30, Proposition 3.7].

As an application of Remark 3.14 (40) and Corollary 3.16 (45), we give another proof of the following theorem which is a part of recent deep works by Kh. Hessami Pilehrood, T. Hessami Pilehrood, and Tauraso. The original proof is based on the identity [20, Theorem 2.2] which is different from our identities in Subsection 2.2.

Theorem 3.18 ([20, Theorem 4.3]). *Let k_1 and k_2 be positive integers satisfying the condition that $k_1 + k_2$ is even. Then we have*

$$(46) \quad \zeta_{\mathcal{A}_2}(\{1\}^{k_1-1}, 2, \{1\}^{k_2-1}) = \frac{1}{2} \left\{ 1 - (-1)^{k_2} \binom{k_1 + k_2 + 1}{k_1 + 1} \right\} \frac{B_{\mathbf{p}-k_1-k_2-1}}{k_1 + k_2 + 1} \mathbf{p},$$

$$(47) \quad \zeta_{\mathcal{A}_2}^*(\{1\}^{k_1-1}, 2, \{1\}^{k_2-1}) = \frac{1}{2} \left\{ 1 - (-1)^{k_2} \binom{k_1 + k_2 + 1}{k_2 + 1} \right\} \frac{B_{\mathbf{p}-k_1-k_2-1}}{k_1 + k_2 + 1} \mathbf{p}.$$

Remark 3.19. They also calculated $\zeta_{\mathcal{A}}^\bullet(\{2\}^{k_1}, 3, \{2\}^{k_2})$ and $\zeta_{\mathcal{A}}^\bullet(\{2\}^{k_1}, 1, \{2\}^{k_2})$ where k_1 and k_2 are positive integers and $\bullet \in \{\emptyset, \star\}$ ([20, Theorem 4.1 and Theorem 4.2]).

Proof of Theorem 3.18. Let k_1 and k_2 be positive integers such that $k_1 + k_2$ is even. Let $w := k_1 + k_2 + 1$. First, we show the star case. By Remark 3.14 (40), Proposition 3.7 (26), (28), and (29), we have

$$\begin{aligned} \zeta_{\mathcal{A}_2}^*(\{1\}^{k_1-1}, 2, \{1\}^{k_2-1}) &= -\zeta_{\mathcal{A}_2}^*(k_1, k_2) - (\zeta_{\mathcal{A}_2}^*(1, k_1, k_2) - \zeta_{\mathcal{A}_2}^*(k_1 + 1, k_2)) \mathbf{p} \\ &= -\frac{1}{2} \left\{ (-1)^{k_2} k_1 \binom{w}{k_2} - (-1)^{k_1} k_2 \binom{w}{k_1} - w + 1 \right\} \frac{B_{\mathbf{p}-w}}{w} \mathbf{p} \\ &\quad - \left(-\frac{1}{2} \left\{ (-1)^{k_2} \binom{w}{k_2} + w \right\} \frac{B_{\mathbf{p}-w}}{w} - (-1)^{k_1+1} \binom{w}{k_1+1} \frac{B_{\mathbf{p}-w}}{w} \right) \mathbf{p} \\ &= \frac{1}{2} \left\{ 1 - (-1)^{k_2} \binom{w}{k_2+1} \right\} \frac{B_{\mathbf{p}-w}}{w} \mathbf{p}. \end{aligned}$$

Let $\mathbb{k} = (\{1\}^{k_1-1}, 2, \{1\}^{k_2-1}) =: (l_1, \dots, l_{w-2})$. By Corollary 3.16 (45), we have

$$(48) \quad \zeta_{\mathcal{A}_2}(\mathbb{k}) = \zeta_{\mathcal{A}_2}^*(\overline{\mathbb{k}}) + \sum_{j=1}^{w-3} (-1)^j \zeta_{\mathcal{A}_2}(l_1, \dots, l_j) \zeta_{\mathcal{A}_2}^*(l_{w-2}, \dots, l_{j+1}).$$

We see that one of the following cases is satisfied for $j = 1, \dots, w-3$:

- (i) At least one of $\zeta_{\mathcal{A}_2}(l_1, \dots, l_j)$ and $\zeta_{\mathcal{A}_2}^*(l_{w-2}, \dots, l_{j+1})$ is zero,
- (ii) Both of $\zeta_{\mathcal{A}_2}(l_1, \dots, l_j)$ and $\zeta_{\mathcal{A}_2}^*(l_{w-2}, \dots, l_{j+1})$ belong to $\mathbf{p}\mathcal{A}_2$.

Therefore, the summation in the equality (48) vanishes and we have

$$\zeta_{\mathcal{A}_2}(\mathbb{k}) = \zeta_{\mathcal{A}_2}^*(\overline{\mathbb{k}}).$$

This completes the proof. \square

3.3. Functional equations for the index $\{1\}^m$. In this subsection, we argue about functional equations of FMPs of the index $\{1\}^m$.

Lemma 3.20. *Let m be a positive integer. Then*

$$(49) \quad \widetilde{\mathcal{L}}_{\mathcal{A}, \{1\}^m}^*(t) = \mathcal{L}_{\mathcal{A}, m}(1-t),$$

$$(50) \quad \mathcal{L}_{\mathcal{A}, \{1\}^m}(t) = (-1)^{m-1} \mathcal{L}_{\mathcal{A}, m}(1-t).$$

Proof. By Proposition 3.7 (23), the cases $\mathbb{k} = \{1\}^m$ in theorem 1.3 (1) and (2) give the equalities (49) and (50), respectively. \square

By Lemma 3.20 and Proposition 3.11 (36), we can express every FMP of the index $\{1\}^m$ by a FP. Therefore, we can obtain functional equations of FMPs of the index $\{1\}^m$ from functional equations of FPs. For example, we get distribution properties for FMPs of the index $\{1\}^m$ by the following result by Elbaz-Vincent and Gangl:

Proposition 3.21 (Elbaz-Vincent and Gangl [4, Proposition 5.7 (2)]). *Let n be a non-zero integer and m a positive integer. Let ζ_n be a primitive $|n|$ -th root of unity. Then we have the following equality in $\mathcal{A}_{\mathbb{Z}[\zeta_n, t]}$:*

$$(51) \quad \mathcal{L}_{\mathcal{A}, m}(t^n) = n^{m-1} \sum_{k=0}^{|n|-1} \frac{1 - t^{n\mathbf{p}}}{1 - (\zeta_n^k t)^{\mathbf{p}}} \mathcal{L}_{\mathcal{A}, m}(\zeta_n^k t).$$

Theorem 3.22 (Distribution properties for FMPs of the index $\{1\}^m$). *Let n be a non-zero integer and m a positive integer. Let ζ_n be a primitive $|n|$ -th root of unity. Then the following equalities hold in $\mathcal{A}_{\mathbb{Z}[\zeta_n][t]}$:*

$$(52) \quad \mathcal{L}_{\mathcal{A}, \{1\}^m}(1 - t^n) = n^{m-1} \sum_{k=0}^{|n|-1} \frac{1 - t^{n\mathbf{p}}}{1 - (\zeta_n^k t)^{\mathbf{p}}} \mathcal{L}_{\mathcal{A}, \{1\}^m}(1 - \zeta_n^k t),$$

$$(53) \quad \widetilde{\mathcal{L}}_{\mathcal{A}, \{1\}^m} \left(\frac{1}{1 - t^n} \right) = n^{m-1} \sum_{k=0}^{|n|-1} \widetilde{\mathcal{L}}_{\mathcal{A}, \{1\}^m} \left(\frac{1}{1 - \zeta_n^k t} \right),$$

$$(54) \quad \mathcal{L}_{\mathcal{A}, \{1\}^m}^* \left(\frac{1}{1 - t^n} \right) = n^{m-1} \sum_{k=0}^{|n|-1} \mathcal{L}_{\mathcal{A}, \{1\}^m}^* \left(\frac{1}{1 - \zeta_n^k t} \right),$$

$$(55) \quad \widetilde{\mathcal{L}}_{\mathcal{A}, \{1\}^m}^*(1 - t^n) = n^{m-1} \sum_{k=0}^{|n|-1} \frac{1 - t^{n\mathbf{p}}}{1 - (\zeta_n^k t)^{\mathbf{p}}} \widetilde{\mathcal{L}}_{\mathcal{A}, \{1\}^m}^*(1 - \zeta_n^k t).$$

Proof. These are obtained by Proposition 3.21. Note that $(1 - t^n)^{\mathbf{p}} = 1 - t^{n\mathbf{p}}$ and $(1 - \zeta_n^k t)^{\mathbf{p}} = 1 - (\zeta_n^k t)^{\mathbf{p}}$ in $\mathcal{A}_{\mathbb{Z}[\zeta_n, t]}$. \square

Corollary 3.23. *Let m be a positive integer. Then the following equalities hold in $\mathcal{A}_{\mathbb{Z}[t]}$:*

$$(56) \quad \tilde{\mathcal{L}}_{\mathcal{A}, \{1\}^m}(t) = (-1)^{m-1} \tilde{\mathcal{L}}_{\mathcal{A}, \{1\}^m}(1-t),$$

$$(57) \quad \mathcal{L}_{\mathcal{A}, \{1\}^m}^*(t) = (-1)^{m-1} \mathcal{L}_{\mathcal{A}, \{1\}^m}^*(1-t).$$

Proof. Let $n = -1$ in Theorem 3.22. Then we have the desired formulas by replacing $1/(1-t)$ with t . \square

Remark 3.24. Lemma 3.20 (50) has been proved by Mattarei and Tauraso ([27, The proof of Theorem 2.3], [18, Lemma 3.2]) and Lemma 3.23 (56) has been proved by L. L. Zhao and Z. W. Sun ([32, Theorem 1.2]).

4. SPECIAL VALUES OF FINITE MULTIPLE POLYLOGARITHMS

4.1. Special values of F(S)MPs. We calculate some special values of F(S)MPs in \mathcal{A} and \mathcal{A}_2 by applying our main results.

Lemma 4.1 (Tauraso and J. Zhao [28]). *Let m be an integer greater than 1. Let k_1 and k_2 be positive integers such that $w := k_1 + k_2$ is odd. Then we have the following equalities:*

$$(58) \quad \mathcal{L}_{\mathcal{A}, m}(-1) = \frac{1 - 2^{m-1}}{2^{m-2}} \frac{B_{\mathbf{p}-m}}{m},$$

$$(59) \quad \mathcal{L}_{\mathcal{A}, (k_1, k_2)}(-1) = \tilde{\mathcal{L}}_{\mathcal{A}, (k_1, k_2)}(-1) = \frac{2^{w-1} - 1}{2^{w-1}} \frac{B_{\mathbf{p}-w}}{w},$$

$$(60) \quad \mathcal{L}_{\mathcal{A}, (k_1, k_2)}^*(-1) = \tilde{\mathcal{L}}_{\mathcal{A}, (k_1, k_2)}^*(-1) = \frac{1 - 2^{w-1}}{2^{w-1}} \frac{B_{\mathbf{p}-w}}{w}.$$

Proof. The equalities (58), (59), and (60) are [28, Corollary 2.3], [28, Theorem 3.1 (17)], and [28, Theorem 3.1 (18)], respectively. \square

Proposition 4.2. *Let m be an integer greater than 1. Let k_1 and k_2 be positive integers such that $w := k_1 + k_2$ is odd. Then we have the following equalities:*

$$(61) \quad \mathcal{L}_{\mathcal{A}_2, m}(-1) = \begin{cases} \frac{m(2^m-1)}{2^m} \widehat{B}_{\mathbf{p}-m-1} \mathbf{p} & \text{if } m \text{ is even,} \\ \frac{2^{m-1}-1}{2^{m-2}} (2\widehat{B}_{\mathbf{p}-m} - \widehat{B}_{2\mathbf{p}-m-1}) & \text{if } m \text{ is odd,} \end{cases}$$

$$(62) \quad \mathcal{L}_{\mathcal{A}_2, (k_1, k_2)}(-1) = \frac{1 - 2^{w-1}}{2^{w-1}} (2\widehat{B}_{\mathbf{p}-w} - \widehat{B}_{2\mathbf{p}-w-1}) + \frac{k_2(1 - 2^{k_1-1})}{2^{k_1-1}} \widehat{B}_{\mathbf{p}-k_1} \widehat{B}_{\mathbf{p}-k_2-1} \mathbf{p},$$

$$(63) \quad \mathcal{L}_{\mathcal{A}_2, (k_1, k_2)}^*(-1) = \frac{2^{w-1} - 1}{2^{w-1}} (2\widehat{B}_{\mathbf{p}-w} - \widehat{B}_{2\mathbf{p}-w-1}) + \frac{k_2(1 - 2^{k_1-1})}{2^{k_1-1}} \widehat{B}_{\mathbf{p}-k_1} \widehat{B}_{\mathbf{p}-k_2-1} \mathbf{p},$$

$$(64) \quad \tilde{\mathcal{L}}_{\mathcal{A}_2, (k_1, k_2)}(-1) = \frac{1 - 2^{w-1}}{2^{w-1}} (2\widehat{B}_{\mathbf{p}-w} - \widehat{B}_{2\mathbf{p}-w-1}) + \frac{k_1(1 - 2^{k_2-1})}{2^{k_2-1}} \widehat{B}_{\mathbf{p}-k_1-1} \widehat{B}_{\mathbf{p}-k_2} \mathbf{p},$$

$$(65) \quad \tilde{\mathcal{L}}_{\mathcal{A}_2, (k_1, k_2)}^*(-1) = \frac{2^{w-1} - 1}{2^{w-1}} (2\widehat{B}_{\mathbf{p}-w} - \widehat{B}_{2\mathbf{p}-w-1}) + \frac{k_1(1 - 2^{k_2-1})}{2^{k_2-1}} \widehat{B}_{\mathbf{p}-k_1-1} \widehat{B}_{\mathbf{p}-k_2} \mathbf{p},$$

where we assume that k_1 (resp. k_2) is greater than 1 in the equalities (62) and (63) (resp. (64) and (65)).

Proof. The equality (61) is obtained by Z. H. Sun's results ([23, Theorem 5.2 (b), Corollary 5.2 (a)]) and the relation

$$(66) \quad \mathcal{L}_{\mathcal{A}_n, m}(-1) = -\zeta_{\mathcal{A}_n}(m) + \frac{1}{2^{m-1}} \left(\sum_{k=1}^{\frac{p-1}{2}} \frac{1}{k^m} \bmod p^n \right),$$

where n is any positive integer. Tauraso and J. Zhao also proved the even case of the equality (61) ([28, Corollary 2.3]). Now, we consider the following relation:

$$\mathcal{L}_{\mathcal{A}_2, k_1}(-1) \zeta_{\mathcal{A}_2}(k_2) = \mathcal{L}_{\mathcal{A}_2, (k_1, k_2)}(-1) + \tilde{\mathcal{L}}_{\mathcal{A}_2, (k_2, k_1)}(-1) + \mathcal{L}_{\mathcal{A}_2, k_1+k_2}(-1).$$

Since $k_1 + k_2$ is odd, we have $\mathcal{L}_{\mathcal{A}_2, (k_1, k_2)}(-1) = \tilde{\mathcal{L}}_{\mathcal{A}_2, (k_2, k_1)}(-1)$ by Proposition 3.11 (36). Therefore, we obtain the equalities (62) and (64) by Proposition 3.7 (23), Lemma 4.1 (58), and the equality (61). The proof of the equalities (63) and (65) is similar. \square

Proposition 4.3. *Let m be an integer greater than 1. Let k_1 and k_2 be integers such that $w := k_1 + k_2$ is odd. Then we have the following equalities:*

$$(67) \quad \mathcal{L}_{\mathcal{A}, \{1\}^m}(2) = \tilde{\mathcal{L}}_{\mathcal{A}, \{1\}^m}^*(2) = \frac{1 - 2^{m-1}}{2^{m-2}} \frac{B_{\mathbf{p}-m}}{m},$$

$$(68) \quad \tilde{\mathcal{L}}_{\mathcal{A}, \{1\}^m}(1/2) = \mathcal{L}_{\mathcal{A}, \{1\}^m}^*(1/2) = \frac{2^{m-1} - 1}{2^{m-1}} \frac{B_{\mathbf{p}-m}}{m},$$

$$(69) \quad \mathcal{L}_{\mathcal{A}, (\{1\}^{k_1-1}, 2, \{1\}^{k_2-1})}(2) = \left\{ \frac{2^{w-1} - 1}{2^{w-1}} - (-1)^{k_1} \binom{w}{k_1} \right\} \frac{B_{\mathbf{p}-w}}{w},$$

$$(70) \quad \tilde{\mathcal{L}}_{\mathcal{A}, (\{1\}^{k_1-1}, 2, \{1\}^{k_2-1})}^*(2) = \left\{ \frac{1 - 2^{w-1}}{2^{w-1}} - (-1)^{k_1} \binom{w}{k_1} \right\} \frac{B_{\mathbf{p}-w}}{w},$$

$$(71) \quad \tilde{\mathcal{L}}_{\mathcal{A}, (\{1\}^{k_1-1}, 2, \{1\}^{k_2-1})}(1/2) = \frac{1}{2} \left\{ \frac{1 - 2^{w-1}}{2^{w-1}} - (-1)^{k_1} \binom{w}{k_1} \right\} \frac{B_{\mathbf{p}-w}}{w},$$

$$(72) \quad \mathcal{L}_{\mathcal{A}, (\{1\}^{k_1-1}, 2, \{1\}^{k_2-1})}^*(1/2) = \frac{1}{2} \left\{ \frac{2^{w-1} - 1}{2^{w-1}} - (-1)^{k_1} \binom{w}{k_1} \right\} \frac{B_{\mathbf{p}-w}}{w}.$$

Proof. First, we prove the star cases. We use the functional equation (1) for an index \mathbb{k}^\vee :

$$(73) \quad \tilde{\mathcal{L}}_{\mathcal{A}, \mathbb{k}^\vee}^*(t) = \tilde{\mathcal{L}}_{\mathcal{A}, \mathbb{k}}^*(1-t) - \zeta_{\mathcal{A}}^*(\mathbb{k}).$$

Consider the case $t = 2$ and $\mathbb{k} = m$ of the equality (73) (or the equality (49)). Then we obtain the star case of the equality (67) by Lemma 4.1 (58). Considering the case $\mathbb{k} = (k_1, k_2)$ of the equality (73), we obtain the equality (70) by Lemma 4.1 (59) and Proposition 3.7 (26). The star case of the equality (68) and the equality (72) are obtained by Proposition 3.11.

Next, we prove the non-star cases by Corollary 3.16 (43) for \mathcal{A} . By considering the case $t = 2$ and $\mathbb{k} = \{1\}^m$ (i.e. the case $t = 2$ of the equality (50)), we have the non-star case of the equality (67). We consider the case $\mathbb{k} = (\{1\}^{k_1-1}, 2, \{1\}^{k_2-1})$ satisfying the condition that $k_1 + k_2$ is odd. The summation for $j = k_1, \dots, w-2$ of the right hand side of (43) vanishes since $\zeta_{\mathcal{A}}^*(\{1\}^{w-1-j}) = 0$. We suppose that j is an element of $\{0, \dots, k_1-1\}$. If j is odd, we have $\zeta_{\mathcal{A}}^*(\{1\}^{k_2-1}, 2, \{1\}^{k_1-j-1}) = 0$ by Theorem 3.18 and if j is even, we have $\mathcal{L}_{\mathcal{A}, \{1\}^j}(2) = 0$ by the equality (67). Hence, we see that the summation in the right hand

side of (43) vanishes and we have the equality (69). The non-star case of the equality (68) and the equality (71) are obtained by Proposition 3.11. \square

Remark 4.4. Z. W. Sun proved that $\mathfrak{L}_{\mathcal{A},\{1\}^2}^*(1/2) = 0$ (see [25, Theorem 1.1]). The proof is based on some technical calculations. The case $(k_1, k_2) = (1, 2)$ or $(k_1, k_2) = (2, 1)$ of Proposition 4.3 (70) and (69) have already been obtained by Meštrović [16, Theorem 1.1, Corollary 1.2] and by Tauraso and J. Zhao [28, Proposition 7.1].

Definition 4.5. Let n be a positive integer and a a non-zero rational number. We define the element $q_{\mathbf{p}}(a)$ of \mathcal{A}_n to be $(q_p(a) \bmod p^n)_p$ where $q_p(a)$ is the Fermat quotient, that is,

$$q_p(a) = \frac{a^{p-1} - 1}{p}$$

for a prime number p .

By Fermat's little theorem, $q_{\mathbf{p}}(a)$ is well-defined as an element of \mathcal{A}_n . Under the hypothesis that *abc*-conjecture is true, we see that $q_{\mathbf{p}}(a)$ is non-zero. See [22].

Theorem 4.6. *Let m be a positive even number. Then we have the following equalities in \mathcal{A}_2 :*

$$(74) \quad \mathfrak{L}_{\mathcal{A}_2, \{1\}^m}(2) = -\tilde{\mathfrak{L}}_{\mathcal{A}_2, \{1\}^m}^*(2) = \left(\frac{m+1}{2^m} - m - 2 \right) \frac{B_{\mathbf{p}-m-1}}{m+1} \mathbf{p},$$

$$(75) \quad \tilde{\mathfrak{L}}_{\mathcal{A}_2, \{1\}^m}(1/2) = -\mathfrak{L}_{\mathcal{A}_2, \{1\}^m}^*(1/2) = \frac{1 - 2^{m+1}}{2^{m+1}} \frac{B_{\mathbf{p}-m-1}}{m+1} \mathbf{p}.$$

Proof. First, we prove the star cases. By the functional equation Remark 3.14 (39), we have

$$\tilde{\mathfrak{L}}_{\mathcal{A}_2, \{1\}^m}^*(2) - \zeta_{\mathcal{A}_2}^*(\{1\}^m) = \mathfrak{L}_{\mathcal{A}_2, m}(-1) + (\tilde{\mathfrak{L}}_{\mathcal{A}_2, (1, m)}^*(-1) - \mathfrak{L}_{\mathcal{A}_2, m+1}(-1)) \mathbf{p}.$$

Therefore, by combining Proposition 3.7 (23), Lemma 4.1 (58), (59) and Proposition 4.2 (61), we have

$$\tilde{\mathfrak{L}}_{\mathcal{A}_2, \{1\}^m}^*(2) - \frac{B_{\mathbf{p}-m-1}}{m+1} \mathbf{p} = \frac{m(2^m - 1)}{2^m} \frac{B_{\mathbf{p}-m-1}}{m+1} \mathbf{p} + \left(\frac{1 - 2^m}{2^m} \frac{B_{\mathbf{p}-m-1}}{m+1} - \frac{1 - 2^m}{2^{m-1}} \frac{B_{\mathbf{p}-m-1}}{m+1} \right) \mathbf{p}$$

or

$$(76) \quad \tilde{\mathfrak{L}}_{\mathcal{A}_2, \{1\}^m}^*(2) = \left(m + 2 - \frac{m+1}{2^m} \right) \frac{B_{\mathbf{p}-m-1}}{m+1} \mathbf{p}.$$

By the equality (34), we have

$$\mathfrak{L}_{\mathcal{A}_2, \{1\}^m}^*(1/2) = \frac{1}{2^{\mathbf{p}}} \left(\tilde{\mathfrak{L}}_{\mathcal{A}_2, \{1\}^m}^*(2) + \mathbf{p} \sum_{i=1}^m \tilde{\mathfrak{L}}_{\mathcal{A}_2, (\{1\}^{i-1}, 2, \{1\}^{m-i})}^*(2) \right).$$

Hence, by combining the equality (76) and Proposition 4.3 (70), we have

$$\begin{aligned} \mathfrak{L}_{\mathcal{A}_2, \{1\}^m}^*(1/2) &= \frac{1}{2} \left\{ \left(m + 2 - \frac{m+1}{2^m} \right) \frac{B_{\mathbf{p}-m-1}}{m+1} \mathbf{p} + \sum_{i=1}^m \left\{ \frac{1 - 2^m}{2^m} - (-1)^i \binom{m+1}{i} \right\} \frac{B_{\mathbf{p}-m-1}}{m+1} \mathbf{p} \right\} \\ &= \frac{2^{m+1} - 1}{2^{m+1}} \frac{B_{\mathbf{p}-m-1}}{m+1} \mathbf{p}, \end{aligned}$$

since m is even and $\sum_{i=1}^m (-1)^i \binom{m+1}{i} = 0$. Note that the equality $2^{\mathbf{p}} = 2(1 + q_{\mathbf{p}}(2)^{\mathbf{p}})$ holds in \mathcal{A}_2 and $\mathfrak{L}_{\mathcal{A}, \{1\}^m}^*(1/2) = 0$.

Next, we prove the non-star cases. By Corollary 3.16 (43) for \mathcal{A}_2 , we have

$$(77) \quad \mathfrak{L}_{\mathcal{A}_2, \{1\}^m}(2) = -\widetilde{\mathfrak{L}}_{\mathcal{A}_2, \{1\}^m}^*(2) + \sum_{j=1}^{m-1} (-1)^{j-1} \mathfrak{L}_{\mathcal{A}_2, \{1\}^j}(2) \zeta_{\mathcal{A}_2}^*(\{1\}^{m-j}).$$

Since $\zeta_{\mathcal{A}_2}^*(\{1\}^{m-j})$ is contained in $\mathbf{p}\mathcal{A}_2$, we have

$$\mathfrak{L}_{\mathcal{A}_2, \{1\}^j}(2) \zeta_{\mathcal{A}_2}^*(\{1\}^{m-j}) = (\text{a certain rational number}) \times B_{\mathbf{p}-m+j-1} B_{\mathbf{p}-j} \mathbf{p}$$

for any $j = 1, \dots, m-1$ by Proposition 3.7 (23) and Proposition 4.3 (67). If j is odd, we have $B_{\mathbf{p}-m+j-1} = 0$ and if j is even, we have $B_{\mathbf{p}-j} = 0$ because $B_{2n+1} = 0$ for any positive integer n . Therefore, the summation in the right hand side of (77) vanishes and we have

$$(78) \quad \mathfrak{L}_{\mathcal{A}_2, \{1\}^m}(2) = -\widetilde{\mathfrak{L}}_{\mathcal{A}_2, \{1\}^m}^*(2) = \left(\frac{m+1}{2^m} - m - 2 \right) \frac{B_{\mathbf{p}-m-1}}{m+1} \mathbf{p}.$$

by Proposition 4.3 (67) and (69), Proposition 3.7 (23), and Proposition 4.2 (61). By the equality (34), the equality (78), and Proposition 4.3 (69), we also have

$$\begin{aligned} \widetilde{\mathfrak{L}}_{\mathcal{A}_2, \{1\}^m}(1/2) &= \frac{1}{2^{\mathbf{p}}} (\mathfrak{L}_{\mathcal{A}_2, \{1\}^m}(2) + \mathbf{p} \sum_{i=1}^m \mathfrak{L}_{\mathcal{A}_2, (\{1\}^{i-1}, 2, \{1\}^{m-i})}(2)) \\ &= \frac{1}{2} \left\{ \left(\frac{m+1}{2^m} - m - 2 \right) \frac{B_{\mathbf{p}-m-1}}{m+1} \mathbf{p} + \sum_{i=1}^m \left\{ \frac{2^m - 1}{2^m} - (-1)^i \binom{m+1}{i} \right\} \frac{B_{\mathbf{p}-m-1}}{m+1} \mathbf{p} \right\} \\ &= \frac{1 - 2^{m+1}}{2^{m+1}} \frac{B_{\mathbf{p}-m-1}}{m+1} \mathbf{p}. \end{aligned}$$

□

Remark 4.7. The cases $m = 2$ of Theorem 4.6 have already been given by Z. W. Sun and L. L. Zhao [26], Meštrović [16], and Tauraso and J. Zhao [28]. Indeed, the non-star case of the equality (74) is [28, Proposition 7.1 (78)] and the star case of the equality (74) which is equivalent to Proposition 4.9 (88) below is [16, Theorem 1.1 (1)] or [28, Proposition 7.1(77)]. The star case of Theorem 4.6 (75) was conjectured by Z. W. Sun [31, Conjecture 1.1] and proved by Z. W. Sun and L. L. Zhao [26]. Meštrović gave another proof of Sun's conjecture in [16] and our proof of the equality (75) is similar to his proof.

Now, we recall the following results for the finite polylogarithms obtained by Z. H. Sun [23, 24], Dilcher and Skula [3], and Meštrović [17]:

Lemma 4.8. *The following equalities hold:*

$$\begin{aligned}
(79) \quad \mathcal{L}_{\mathcal{A}_3,1}(-1) &= -2q_{\mathbf{p}}(2) + q_{\mathbf{p}}(2)^2 \mathbf{p} - \left(\frac{2}{3}q_{\mathbf{p}}(2)^3 + \frac{1}{4}B_{\mathbf{p}-3} \right) \mathbf{p}^2, \\
(80) \quad \mathcal{L}_{\mathcal{A}_3,1}(2) &= -2q_{\mathbf{p}}(2) - \frac{7}{12}B_{\mathbf{p}-3}\mathbf{p}^2, \\
(81) \quad \mathcal{L}_{\mathcal{A}_2,2}(2) &= -q_{\mathbf{p}}(2)^2 + \left(\frac{2}{3}q_{\mathbf{p}}(2)^3 + \frac{7}{6}B_{\mathbf{p}-3} \right) \mathbf{p}, \\
(82) \quad \mathcal{L}_{\mathcal{A}_3,3}(2) &= -\frac{1}{3}q_{\mathbf{p}}(2)^3 - \frac{7}{24}B_{\mathbf{p}-3}, \\
(83) \quad \mathcal{L}_{\mathcal{A}_3,1}(1/2) &= q_{\mathbf{p}}(2) - \frac{1}{2}q_{\mathbf{p}}(2)^2 \mathbf{p} + \left(\frac{1}{3}q_{\mathbf{p}}(2)^3 - \frac{7}{48}B_{\mathbf{p}-3} \right) \mathbf{p}^2, \\
(84) \quad \mathcal{L}_{\mathcal{A}_2,2}(1/2) &= -\frac{1}{2}q_{\mathbf{p}}(2)^2 + \left(\frac{1}{2}q_{\mathbf{p}}(2)^3 + \frac{7}{24}B_{\mathbf{p}-3} \right) \mathbf{p}, \\
(85) \quad \mathcal{L}_{\mathcal{A}_3,3}(1/2) &= \frac{1}{6}q_{\mathbf{p}}(2)^3 + \frac{7}{48}B_{\mathbf{p}-3}.
\end{aligned}$$

Proof. The equality (79) is obtained by [23, Theorem 5.2 (c)] and the equality (66). The equalities (80) and (81) are [24, Theorem 4.1 (i)] and [24, Theorem 4.1 (ii)], respectively. The equalities (82), (83), and (84) are essentially due to Dilcher and Skula [3] (see [24, Remark 4.1]). The equality (83) is also shown by Meštrović [17]. The equality (85) is obtained by the equality (81) and Proposition 3.11. \square

We obtain the following special values for F(S)MP by the above lemma:

Proposition 4.9. *Let $\bullet \in \{\emptyset, \star\}$. Then the following equalities hold:*

$$\begin{aligned}
(86) \quad \mathcal{L}_{\mathcal{A}_2,\{1\}^2}(-1) &= -\tilde{\mathcal{L}}_{\mathcal{A}_2,\{1\}^2}^{\star}(-1) = q_{\mathbf{p}}(2)^2 - \left(q_{\mathbf{p}}(2)^3 + \frac{13}{24}B_{\mathbf{p}-3} \right) \mathbf{p}, \\
(87) \quad \mathcal{L}_{\mathcal{A}_2,\{1\}^2}^{\star}(-1) &= -\tilde{\mathcal{L}}_{\mathcal{A}_2,\{1\}^2}(-1) = q_{\mathbf{p}}(2)^2 - \left(q_{\mathbf{p}}(2)^3 + \frac{1}{24}B_{\mathbf{p}-3} \right) \mathbf{p}, \\
(88) \quad \mathcal{L}_{\mathcal{A}_2,\{1\}^2}(2) &= -\tilde{\mathcal{L}}_{\mathcal{A}_2,\{1\}^2}(2) = -q_{\mathbf{p}}(2)^2 + \left(\frac{2}{3}q_{\mathbf{p}}(2)^3 + \frac{1}{12}B_{\mathbf{p}-3} \right) \mathbf{p}, \\
(89) \quad \mathcal{L}_{\mathcal{A}_2,\{1\}^2}(1/2) &= -\tilde{\mathcal{L}}_{\mathcal{A}_2,\{1\}^2}^{\star}(1/2) = \frac{1}{2}q_{\mathbf{p}}(2)^2 - \frac{1}{2}q_{\mathbf{p}}(2)^3 \mathbf{p}, \\
(90) \quad \mathcal{L}_{\mathcal{A},(1,2)}^{\star}(2) &= -\tilde{\mathcal{L}}_{\mathcal{A},(2,1)}(2) = -\frac{1}{3}q_{\mathbf{p}}(2)^3 - \frac{25}{24}B_{\mathbf{p}-3}, \\
(91) \quad \mathcal{L}_{\mathcal{A},(1,2)}(1/2) &= -\tilde{\mathcal{L}}_{\mathcal{A},(2,1)}^{\star}(1/2) = -\frac{1}{6}q_{\mathbf{p}}(2)^3 - \frac{25}{48}B_{\mathbf{p}-3}, \\
(92) \quad \mathcal{L}_{\mathcal{A},(2,1)}^{\star}(2) &= -\tilde{\mathcal{L}}_{\mathcal{A},(1,2)}(2) = -\frac{1}{3}q_{\mathbf{p}}(2)^3 + \frac{23}{24}B_{\mathbf{p}-3},
\end{aligned}$$

$$(93) \quad \mathfrak{L}_{\mathcal{A},(2,1)}(1/2) = -\tilde{\mathfrak{L}}_{\mathcal{A},(1,2)}^*(1/2) = -\frac{1}{6}q_{\mathbf{p}}(2)^3 + \frac{23}{48}B_{\mathbf{p}-3},$$

$$(94) \quad \mathfrak{L}_{\mathcal{A},\{1\}^3}^\bullet(-1) = \tilde{\mathfrak{L}}_{\mathcal{A},\{1\}^3}^\bullet(-1) = \mathfrak{L}_{\mathcal{A},\{1\}^3}^*(2) = \tilde{\mathfrak{L}}_{\mathcal{A},\{1\}^3}(2) = -\frac{1}{3}q_{\mathbf{p}}(2)^3 - \frac{7}{24}B_{\mathbf{p}-3},$$

$$(95) \quad \mathfrak{L}_{\mathcal{A},\{1\}^3}(1/2) = \tilde{\mathfrak{L}}_{\mathcal{A},\{1\}^3}^*(1/2) = \frac{1}{6}q_{\mathbf{p}}(2)^3 + \frac{7}{48}B_{\mathbf{p}-3}.$$

Proof. We can calculate $\tilde{\mathfrak{L}}_{\mathcal{A}_2,\{1\}^2}^*(-1)$, $\mathfrak{L}_{\mathcal{A}_2,\{1\}^2}^*(2)$, $\mathfrak{L}_{\mathcal{A},(1,2)}^*(2)$, $\mathfrak{L}_{\mathcal{A},(2,1)}^*(2)$, $\tilde{\mathfrak{L}}_{\mathcal{A},\{1\}^3}^*(-1)$, and $\mathfrak{L}_{\mathcal{A},\{1\}^3}^*(2)$ by the equalities

$$\tilde{\mathfrak{L}}_{\mathcal{A}_2,\{1\}^2}^*(-1) = \zeta_{\mathcal{A}_2}^*(\{1\}^2) + \mathfrak{L}_{\mathcal{A}_2,2}(2) + (\tilde{\mathfrak{L}}_{\mathcal{A}_2,(1,2)}^*(2) - \mathfrak{L}_{\mathcal{A}_2,3}(2))\mathbf{p},$$

$$\mathfrak{L}_{\mathcal{A}_2,\{1\}^2}^*(2) = \mathfrak{L}_{\mathcal{A}_2,\{1\}^2}(2) + \mathfrak{L}_{\mathcal{A}_2,2}(2), \quad \mathfrak{L}_{\mathcal{A},(1,2)}^*(2) = \mathfrak{L}_{\mathcal{A},(1,2)}(2) + \mathfrak{L}_{\mathcal{A},3}(2),$$

$$\mathfrak{L}_{\mathcal{A},(2,1)}^*(2) = \mathfrak{L}_{\mathcal{A},(2,1)}(2) + \mathfrak{L}_{\mathcal{A},3}(2), \quad \tilde{\mathfrak{L}}_{\mathcal{A},\{1\}^3}^*(-1) = \mathfrak{L}_{\mathcal{A},3}(2), \quad \mathfrak{L}_{\mathcal{A},\{1\}^3}^*(2) = \mathfrak{L}_{\mathcal{A},\{1\}^3}^*(-1),$$

respectively. Here, we have used Remark 3.14 (39), Lemma 3.20 (49), and Corollary 3.23 (57). All other values obtained by Proposition 3.11 and Corollary 3.16 (43) and (44). \square

Remark 4.10. Note that all of the values that appear in the above proposition essentially have been given by Meštrović [16, Theorem 1.1] and Tauraso and J. Zhao [28, Proposition 7.1]. We have determined all values of the form $\tilde{\mathfrak{L}}_{\mathcal{A}_n,\mathbb{k}}^\bullet(r)$ for $- \in \{\emptyset, \sim\}$, $\bullet \in \{\emptyset, \star\}$, and $r \in \{-1, 2^{\pm 1}\}$ when $n + \text{wt}(\mathbb{k}) \leq 4$ by Lemma 4.1, Proposition 4.2, Proposition 4.3, Theorem 4.6, Lemma 4.8, and Proposition 4.9.

Furthermore, we have the following some special values of FMPs of weight 4.

Proposition 4.11. *Let $\bullet \in \{\emptyset, \star\}$. Then the following equalities hold:*

$$(96) \quad \mathfrak{L}_{\mathcal{A},(1,3)}^\bullet(-1) = -\tilde{\mathfrak{L}}_{\mathcal{A},(3,1)}^\bullet(-1) = \frac{1}{2}q_{\mathbf{p}}(2)B_{\mathbf{p}-3},$$

$$(97) \quad \mathfrak{L}_{\mathcal{A},(2,1,1)}(2) = \tilde{\mathfrak{L}}_{\mathcal{A},(1,1,2)}^*(2) = -\frac{1}{2}q_{\mathbf{p}}(2)B_{\mathbf{p}-3},$$

$$(98) \quad \mathfrak{L}_{\mathcal{A},(2,1,1)}^*(1/2) = \tilde{\mathfrak{L}}_{\mathcal{A},(1,1,2)}(1/2) = -\frac{1}{4}q_{\mathbf{p}}(2)B_{\mathbf{p}-3}.$$

Proof. By [28, Proposition 6.1 (55)], we have $\tilde{\mathfrak{L}}_{\mathcal{A},(3,1)}(-1) = -\frac{1}{2}q_{\mathbf{p}}(2)B_{\mathbf{p}-3}$. All of the other values are obtained by the functional equations and Proposition 3.11. \square

Next, we calculate some special values of FH(S)MPs. The following two lemmas are due to Chamberland and Dilcher [1], Tauraso and J. Zhao [28], and Kh. Hessami Pilehrood, T. Hessami Pilehrood, and Tauraso [20]:

Lemma 4.12. *Let m , k_1 , k_2 , and k_3 be positive integers, $w = k_1 + k_2 + k_3$, and $\bullet \in \{\emptyset, \star\}$. Then the following equalities hold:*

$$(99) \quad \mathfrak{L}_{\mathcal{A},(k_1,k_2)}^{*,\bullet}(-1, -1) = (-1)^{k_1} \frac{1 - 2^{k_1+k_2-1}}{2^{k_1+k_2-1}} \binom{k_1+k_2}{k_1} \frac{B_{\mathbf{p}-k_1-k_2}}{k_1+k_2}$$

if $k_1 + k_2$ is odd,

$$(100) \quad \mathfrak{L}_{\mathcal{A},(k_1,k_2)}^{*,\bullet}(-1, -1) = \frac{(2^{k_1-1} - 1)(2^{k_2-1} - 1)}{2^{k_1+k_2-3}k_1k_2} B_{\mathbf{p}-k_1} B_{\mathbf{p}-k_2}$$

if $k_1 + k_2$ is even and $k_1, k_2 \geq 2$,

$$(101) \quad \mathfrak{L}_{\mathcal{A},(m,1)}^{*,\bullet}(-1, -1) = \mathfrak{L}_{\mathcal{A},(1,m)}^{*,\bullet}(-1, -1) = \frac{2^{m-1} - 1}{2^{m-2}m} q_{\mathbf{p}}(2) B_{\mathbf{p}-m}$$

if m is odd and $m \geq 3$,

$$(102) \quad \mathfrak{L}_{\mathcal{A},(k_1,k_2,k_3)}^*(-1, -1, 1) = -\mathfrak{L}_{\mathcal{A},(k_1,k_2,k_3)}^{*,*}(-1, -1, 1) = \frac{1}{2} \left\{ (-1)^{k_3} \binom{w}{k_3} - \frac{1-2^{w-1}}{2^{w-1}} \binom{w}{k_1} \right\} \frac{B_{\mathbf{p}-w}}{w}$$

if k_1 is even and $k_2 + k_3$ odd,

$$(103) \quad -\mathfrak{L}_{\mathcal{A},(k_1,k_2,k_3)}^*(1, -1, -1) = \mathfrak{L}_{\mathcal{A},(k_1,k_2,k_3)}^{*,*}(1, -1, -1) = \frac{1}{2} \left\{ (-1)^{k_1} \binom{w}{k_1} - \frac{1-2^{w-1}}{2^{w-1}} \binom{w}{k_3} \right\} \frac{B_{\mathbf{p}-w}}{w}$$

if $k_1 + k_2$ is odd and k_3 even,

$$(104) \quad \mathfrak{L}_{\mathcal{A},(k_1,k_2,k_3)}^*(-1, 1, -1) = -\mathfrak{L}_{\mathcal{A},(k_1,k_2,k_3)}^{*,*}(-1, 1, -1) = \frac{1-2^{w-1}}{2^w} \left\{ \binom{w}{k_3} - \binom{w}{k_1} \right\} \frac{B_{\mathbf{p}-w}}{w}$$

if k_1 is even, k_2 odd, and k_3 even,

$$(105) \quad \mathfrak{L}_{\mathcal{A},\{1\}^3}^*(1, -1, -1) = -\mathfrak{L}_{\mathcal{A},\{1\}^3}^*(-1, -1, 1) = q_{\mathbf{p}}(2)^3 + \frac{7}{8} B_{\mathbf{p}-3},$$

$$(106) \quad \mathfrak{L}_{\mathcal{A},\{1\}^3}^{*,*}(1, -1, -1) = -\mathfrak{L}_{\mathcal{A},\{1\}^3}^{*,*}(-1, -1, 1) = q_{\mathbf{p}}(2)^3 - \frac{7}{8} B_{\mathbf{p}-3},$$

$$(107) \quad \mathfrak{L}_{\mathcal{A},\{1\}^3}^{*,\bullet}(-1, 1, -1) = 0,$$

$$(108) \quad \mathfrak{L}_{\mathcal{A},\{1\}^3}^{*,\bullet}(-1, -1, -1) = -2\mathfrak{L}_{\mathcal{A},\{1\}^3}^{*,\bullet}(1, -1, 1) = -\frac{4}{3} q_{\mathbf{p}}(2)^3 - \frac{1}{6} B_{\mathbf{p}-3}.$$

Proof. The non-star case of the equality (99) is [28, Theorem 3.1 (15)]. The equality (100) and (101) are [28, Theorem 3.1 (20)]. Now, suppose that w is odd. By [28, Theorem 4.1], we have

$$(109) \quad 2\mathfrak{L}_{\mathcal{A},(k_1,k_2,k_3)}^*(-1, -1, 1) = \zeta_{\mathcal{A}}(k_3, k_1 + k_2) + \mathfrak{L}_{\mathcal{A},(k_2+k_3,k_1)}^*(-1, -1) - \mathfrak{L}_{\mathcal{A},k_1}(-1) \tilde{\mathfrak{L}}_{\mathcal{A},(k_3,k_2)}(-1),$$

$$(110) \quad 2\mathfrak{L}_{\mathcal{A},(k_1,k_2,k_3)}^*(-1, 1, -1) = -\mathfrak{L}_{\mathcal{A},k_1}(-1) \mathfrak{L}_{\mathcal{A},(k_3,k_2)}(-1) - \tilde{\mathfrak{L}}_{\mathcal{A},(k_2,k_1)}(-1) \mathfrak{L}_{\mathcal{A},k_3}(-1) \\ + \mathfrak{L}_{\mathcal{A},(k_3,k_1+k_2)}^*(-1, -1) + \mathfrak{L}_{\mathcal{A},(k_2+k_3,k_1)}^*(-1, -1).$$

If k_1 is even, then we have $\mathfrak{L}_{\mathcal{A},k_1}(-1) = 0$ by Lemma 4.1 (58). Therefore, by the equality (109), we can calculate $\mathfrak{L}_{\mathcal{A},(k_1,k_2,k_3)}^*(-1, -1, 1)$, Proposition 3.7 (26), and the equality (99). If k_1 and k_3 are even, then we have $\mathfrak{L}_{\mathcal{A},k_1}(-1) = \mathfrak{L}_{\mathcal{A},k_3}(-1) = 0$ by Lemma 4.1 (58). Therefore, we can calculate $\mathfrak{L}_{\mathcal{A},(k_1,k_2,k_3)}^*(-1, 1, -1)$ by the equalities (110) and (99). The non-star case of the equality (103) is obtained by Proposition 3.11 and the equality (102) and the non-star cases of the equalities (105), (107), and (108) are obtained by [28, Proposition 7.6]. All star cases are obtained by Theorem 3.15 (41). Note that [28, Theorem 3.1 (16)] which is the corresponding formula to the star case of the equality (99) is incorrect. \square

Lemma 4.13. *Let k_1 and k_2 be positive even integers. Then we have*

$$(111) \quad \mathfrak{L}_{\mathcal{A}_2,(k_1,k_2)}^*(-1, -1) = \left\{ \frac{(k_2 - k_1)(2^{k_1+k_2} - 1)}{2^{k_1+k_2+1}(k_1 + k_2 + 2)} \binom{k_1 + k_2 + 2}{k_1 + 1} - \frac{k_1 + k_2}{2} \right\} \frac{B_{\mathbf{p}-k_1-k_2-1}}{k_1 + k_2 + 1} \mathbf{p},$$

$$(112) \quad \mathfrak{L}_{\mathcal{A}_2,(k_1,k_2)}^{*,*}(-1, -1) = \left\{ \frac{(k_2 - k_1)(2^{k_1+k_2} - 1)}{2^{k_1+k_2+1}(k_1 + k_2 + 2)} \binom{k_1 + k_2 + 2}{k_1 + 1} + \frac{k_1 + k_2}{2} \right\} \frac{B_{\mathbf{p}-k_1-k_2-1}}{k_1 + k_2 + 1} \mathbf{p},$$

$$(113) \quad \mathfrak{L}_{\mathcal{A}_2, \{1\}^2}^*(-1, -1) = 2q_{\mathbf{p}}(2)^2 - \left(2q_{\mathbf{p}}(2)^3 + \frac{1}{3}B_{\mathbf{p}-3}\right) \mathbf{p},$$

$$(114) \quad \mathfrak{L}_{\mathcal{A}_2, \{1\}^2}^{*,*}(-1, -1) = 2q_{\mathbf{p}}(2)^2 - \left(2q_{\mathbf{p}}(2)^3 - \frac{1}{3}B_{\mathbf{p}-3}\right) \mathbf{p},$$

$$(115) \quad \mathfrak{L}_{\mathcal{A}_2, \{1\}^3}^*(-1, -1, -1) = -\frac{4}{3}q_{\mathbf{p}}(2)^3 + \widehat{B}_{\mathbf{p}-3} - \frac{1}{2}\widehat{B}_{2\mathbf{p}-4} + 2\left(q_{\mathbf{p}}(2)^4 - q_{\mathbf{p}}(2)\widehat{B}_{\mathbf{p}-3}\right) \mathbf{p},$$

$$(116) \quad \mathfrak{L}_{\mathcal{A}_2, \{1\}^3}^{*,*}(-1, -1, -1) = -\frac{4}{3}q_{\mathbf{p}}(2)^3 + \widehat{B}_{\mathbf{p}-3} - \frac{1}{2}\widehat{B}_{2\mathbf{p}-4} + 2\left(q_{\mathbf{p}}(2)^4 + q_{\mathbf{p}}(2)\widehat{B}_{\mathbf{p}-3}\right) \mathbf{p}.$$

Proof. The equalities (111), (113), and (115) are [20, Lemma 3.1], [28, Proposition 7.3 (100)], and [28, Proposition 7.6 (117)], respectively. The equalities (112), (114), and (116) are obtained by the relations

$$\begin{aligned} \mathfrak{L}_{\mathcal{A}_2, (k_1, k_2)}^{*,*}(-1, -1) &= \mathfrak{L}_{\mathcal{A}_2, (k_1, k_2)}^*(-1, -1) + \zeta_{\mathcal{A}_2}(k_1 + k_2), \\ \mathfrak{L}_{\mathcal{A}_2, \{1\}^2}^{*,*}(-1, -1) &= \mathfrak{L}_{\mathcal{A}_2, \{1\}^2}^*(-1, -1) + \zeta_{\mathcal{A}_2}(2), \end{aligned}$$

and

$$\mathfrak{L}_{\mathcal{A}_2, \{1\}^3}^{*,*}(-1, -1, -1) = \mathfrak{L}_{\mathcal{A}_2, \{1\}^3}^*(-1, -1, -1) + \widetilde{\mathfrak{L}}_{\mathcal{A}_2, (2, 1)}(-1) + \mathfrak{L}_{\mathcal{A}_2, (1, 2)}(-1) + \mathfrak{L}_{\mathcal{A}_2, 3}(-1),$$

respectively. Here, note that

$$\widetilde{\mathfrak{L}}_{\mathcal{A}_2, (2, 1)}(-1) + \mathfrak{L}_{\mathcal{A}_2, (1, 2)}(-1) = -\frac{3}{2}\left(2\widehat{B}_{\mathbf{p}-3} - \widehat{B}_{2\mathbf{p}-4}\right) - \frac{4}{3}q_{\mathbf{p}}(2)B_{\mathbf{p}-3}\mathbf{p}$$

holds by [28, Proposition 7.3 (105) and (106)]. \square

Theorem 4.14. *Let m, k_1, k_2 , and k_3 be positive integers and $\bullet \in \{\emptyset, \star\}$. Let $w = k_1 + k_2$ and $w' = k_1 + k_2 + k_3$. Then we have the following equalities:*

$$(117) \quad \mathfrak{L}_{\mathcal{A}, \{1\}^w}^*(\{1\}^{k_1-1}, 1/2, 2, \{1\}^{k_2-1}) = (-1)^{k_1} \frac{1 - 2^{w-1}}{2^{w-1}} \binom{w}{k_1} \frac{B_{\mathbf{p}-w}}{w}$$

if w is odd,

$$(118) \quad \mathfrak{L}_{\mathcal{A}, \{1\}^w}^{*,*}(\{1\}^{k_1-1}, 2, 1/2, \{1\}^{k_2-1}) = (-1)^{k_1} \frac{2^{w-1} - 1}{2^{w-1}} \binom{w}{k_1} \frac{B_{\mathbf{p}-w}}{w}$$

if w is even,

$$(119) \quad \mathfrak{L}_{\mathcal{A}, \{1\}^w}^*(\{1\}^{k_1-1}, 1/2, 2, \{1\}^{k_2-1}) = 0$$

if w is even,

$$(120) \quad \mathfrak{L}_{\mathcal{A}, \{1\}^w}^{*,*}(\{1\}^{k_1-1}, 2, 1/2, \{1\}^{k_2-1}) = -\frac{(2^{k_1-1} - 1)(2^{k_2-1} - 1)}{2^{w-3}k_1k_2} B_{\mathbf{p}-k_1} B_{\mathbf{p}-k_2}$$

if w is even and $k_1, k_2 \geq 2$,

$$(121) \quad \mathfrak{L}_{\mathcal{A}, \{1\}^{m+1}}^{*,*}(\{1\}^{m-1}, 2, 1/2) = \mathfrak{L}_{\mathcal{A}, \{1\}^{m+1}}^*(2, 1/2, \{1\}^{m-1}) = \frac{1 - 2^{m-1}}{2^{m-2}m} q_{\mathbf{p}}(2) B_{\mathbf{p}-m}$$

if m is odd and $m \geq 3$,

$$(122) \quad \mathfrak{L}_{\mathcal{A}, \{1\}^w}^*(2, \{1\}^{k_1-2}, 1/2, 2, \{1\}^{k_2-1}) = \frac{2^{w-1} - 1}{2^{w-1}} \left\{ \binom{w}{k_1} - 1 \right\} \frac{B_{\mathbf{p}-w}}{w}$$

if $k_1 \geq 3$ is odd and k_2 is even,

$$(123) \quad \mathfrak{L}_{\mathcal{A},\{1\}^m}^*(2, 1/2, 2, \{1\}^{m-3}) = \frac{(1 - 2^{m-1})(m^2 - m + 2)}{2^m} \frac{B_{\mathbf{p}-m}}{m}$$

if $m \geq 3$ is odd,

$$(124) \quad \mathfrak{L}_{\mathcal{A},\{1\}^w}^{*,*}(\{1\}^{k_1-1}, 2, 1/2, \{1\}^{k_2-2}, 2) = \frac{1 - 2^{w-1}}{2^{w-1}} \left\{ 1 + (-1)^{k_1-1} \binom{w}{k_1} \right\} \frac{B_{\mathbf{p}-w}}{w}$$

if $w \geq 3$ is odd and $k_2 \geq 2$,

$$(125) \quad \mathfrak{L}_{\mathcal{A},\{1\}^w}^*(\{1\}^{k_1-1}, 1/2, 2, \{1\}^{k_2-2}, 1/2) = \frac{1 - 2^{w-1}}{2^w} \left\{ \binom{w}{k_1} - 1 \right\} \frac{B_{\mathbf{p}-w}}{w}$$

if k_1 is even and $k_2 \geq 3$ is odd,

$$(126) \quad \mathfrak{L}_{\mathcal{A},\{1\}^m}^*(\{1\}^{m-3}, 1/2, 2, 1/2) = \frac{(2^{m-1} - 1)(m^2 - m + 2)}{2^{m+1}} \frac{B_{\mathbf{p}-m}}{m}$$

if m is odd and $m \geq 3$,

$$(127) \quad \mathfrak{L}_{\mathcal{A},\{1\}^w}^{*,*}(1/2, \{1\}^{k_1-2}, 2, 1/2, \{1\}^{k_2-1}) = \frac{2^{w-1} - 1}{2^w} \left\{ 1 + (-1)^{k_1} \binom{w}{k_1} \right\} \frac{B_{\mathbf{p}-w}}{w}$$

if $w \geq 3$ is odd and $k_1 \geq 2$,

$$(128) \quad \mathfrak{L}_{\mathcal{A},\{1\}^m}^*(1, 2, \{1\}^{m-2}) = -2\mathfrak{L}_{\mathcal{A},\{1\}^m}^*(\{1\}^{m-2}, 1/2, 1) = \frac{(2^{m-1} - 1)(m - 1)}{2^{m-1}} \frac{B_{\mathbf{p}-m}}{m}$$

if $m \geq 3$ is odd,

$$(129) \quad \mathfrak{L}_{\mathcal{A},\{1\}^m}^{*,*}(\{1\}^{m-2}, 2, 1) = -2\mathfrak{L}_{\mathcal{A},\{1\}^m}^{*,*}(1, 1/2, \{1\}^{m-2}) = \frac{(2^{m-1} - 1)(m - 1)}{2^{m-1}} \frac{B_{\mathbf{p}-m}}{m}$$

if $m \geq 3$ is odd,

$$(130) \quad \mathfrak{L}_{\mathcal{A},(\{1\}^{k_1+k_2-1}, 2, \{1\}^{k_3-1})}^*(\{1\}^{k_1-1}, 1/2, 2, \{1\}^{k_2+k_3-2}) = \frac{1}{2} \left\{ (-1)^{k_3} \binom{w'}{k_3} - \frac{1 - 2^{w'-1}}{2^{w'-1}} \binom{w'}{k_1} \right\} \frac{B_{\mathbf{p}-w'}}{w'}$$

if k_1 is even and $k_2 + k_3$ odd,

$$(131) \quad \mathfrak{L}_{\mathcal{A},(\{1\}^{k_1+k_2-1}, 2, \{1\}^{k_3-1})}^{*,*}(\{1\}^{k_1-1}, 2, 1/2, \{1\}^{k_2+k_3-2}) = \frac{1}{2} \left\{ (-1)^{k_3} \binom{w'}{k_3} - \frac{1 - 2^{w'-1}}{2^{w'-1}} \binom{w'}{k_1} \right\} \frac{B_{\mathbf{p}-w'}}{w'}$$

if k_1 is even and $k_2 + k_3$ odd,

$$(132) \quad \mathfrak{L}_{\mathcal{A},(\{1\}^{k_1-1}, 2, \{1\}^{k_2+k_3-1})}^*(\{1\}^{k_1+k_2-2}, 1/2, 2, \{1\}^{k_3-1}) = -\frac{1}{2} \left\{ (-1)^{k_1} \binom{w'}{k_1} - \frac{1 - 2^{w'-1}}{2^{w'-1}} \binom{w'}{k_3} \right\} \frac{B_{\mathbf{p}-w'}}{w'}$$

if $k_1 + k_2$ odd and k_3 even,

$$(133) \quad \mathfrak{L}_{\mathcal{A},(\{1\}^{k_1-1}, 2, \{1\}^{k_2+k_3-1})}^{*,*}(\{1\}^{k_1+k_2-2}, 2, 1/2, \{1\}^{k_3-1}) = -\frac{1}{2} \left\{ (-1)^{k_1} \binom{w'}{k_1} - \frac{1 - 2^{w'-1}}{2^{w'-1}} \binom{w'}{k_3} \right\} \frac{B_{\mathbf{p}-w'}}{w'}$$

if $k_1 + k_2$ odd and k_3 even,

$$(134) \quad \mathfrak{L}_{\mathcal{A},\{1\}^{w'}}^*(\{1\}^{k_1-1}, 1/2, 2, \{1\}^{k_2-2}, 1/2, 2, \{1\}^{k_3-1}) = \frac{1 - 2^{w'-1}}{2^{w'}} \left\{ \binom{w'}{k_1} - \binom{w'}{k_3} \right\} \frac{B_{\mathbf{p}-w'}}{w'}$$

if k_1 is even, k_2 odd, k_3 even, and $k_2 > 1$,

$$(135) \quad \mathfrak{L}_{\mathcal{A},\{1\}^{w'}}^{*,*}(\{1\}^{k_1-1}, 2, 1/2, \{1\}^{k_2-2}, 2, 1/2, \{1\}^{k_3-1}) = \frac{1 - 2^{w'-1}}{2^{w'}} \left\{ \binom{w'}{k_3} - \binom{w'}{k_1} \right\} \frac{B_{\mathbf{p}-w'}}{w'}$$

if k_1 is even, k_2 odd, k_3 even, and $k_2 > 1$,

$$(136) \quad \mathfrak{L}_{\mathcal{A},\{1\}^{w+1}}^*(\{1\}^{k_1-1}, 1/2, 1, 2, \{1\}^{k_2-1}) = \frac{1-2^w}{2^{w+1}} \left\{ \binom{w+1}{k_1} - \binom{w+1}{k_2} \right\} \frac{B_{\mathbf{p}-w-1}}{w+1}$$

if k_1 and k_2 are even,

$$(137) \quad \mathfrak{L}_{\mathcal{A},\{1\}^{w+1}}^{*,*}(\{1\}^{k_1-1}, 2, 1, 1/2, \{1\}^{k_2-1}) = \frac{1-2^w}{2^{w+1}} \left\{ \binom{w+1}{k_2} - \binom{w+1}{k_1} \right\} \frac{B_{\mathbf{p}-w-1}}{w+1}$$

if k_1 and k_2 are even,

$$(138) \quad \mathfrak{L}_{\mathcal{A},(1,2)}^{*,\bullet}(2, 1/2) = -\mathfrak{L}_{\mathcal{A},(2,1)}^{*,\bullet}(2, 1/2) = q_{\mathbf{p}}(2)^3 - \frac{7}{8}B_{\mathbf{p}-3},$$

$$(139) \quad \mathfrak{L}_{\mathcal{A},(1,2)}^{*,\bullet}(1/2, 2) = -\mathfrak{L}_{\mathcal{A},(2,1)}^{*,\bullet}(1/2, 2) = -\frac{7}{8}B_{\mathbf{p}-3},$$

$$(140) \quad \mathfrak{L}_{\mathcal{A},\{1\}^3}^{*,\bullet}(2, 1, 1/2) = 0.$$

Proof. First, we prove the star-cases. By Theorem 3.12, we have

$$(141) \quad \mathfrak{L}_{\mathcal{A},(k_1,k_2)}^{\text{III},*}(s, t) = \mathfrak{L}_{\mathcal{A},\{1\}^{k_1+k_2}}^{\text{III},*}(\{1\}^{k_1-1}, 1-s, \{1\}^{k_2-1}, 1-t) - \mathfrak{L}_{\mathcal{A},\{1\}^{k_1+k_2}}^{\text{III},*}(\{1\}^{k_1-1}, 1-s, \{1\}^{k_2}),$$

where s and t are indeterminates. If we substitute -1 (resp. 1) for s (resp. t) in the equality (141), then we see that

$$(\text{L. H. S. of (141)}) = \mathfrak{L}_{\mathcal{A},(k_1,k_2)}^{\text{III},*}(-1, 1) = \mathfrak{L}_{\mathcal{A},(k_1,k_2)}^{*,*}(-1, -1)$$

and

$$(\text{R. H. S. of (141)}) = -\mathfrak{L}_{\mathcal{A},\{1\}^{k_1+k_2}}^{\text{III},*}(\{1\}^{k_1-1}, 2, \{1\}^{k_2}) = -\mathfrak{L}_{\mathcal{A},\{1\}^{k_1+k_2}}^{*,*}(\{1\}^{k_1-1}, 2, 1/2, \{1\}^{k_2-1}).$$

Therefore, we obtain the equality (118), (120), and (121) by Lemma 4.12 (99), (100), and (101), respectively. Next, if we substitute -1 for s and t in the equality (141), then we see that

$$(\text{L. H. S. of (141)}) = \mathfrak{L}_{\mathcal{A},(k_1,k_2)}^{\text{III},*}(-1, -1) = \mathfrak{L}_{\mathcal{A},(k_1,k_2)}^{*,*}(-1)$$

and

$$(\text{R. H. S. of (141)}) = \mathfrak{L}_{\mathcal{A},\{1\}^{k_1+k_2}}^{\text{III},*}(\{1\}^{k_1-1}, 2, \{1\}^{k_2-1}, 2) - \mathfrak{L}_{\mathcal{A},\{1\}^{k_1+k_2}}^{\text{III},*}(\{1\}^{k_1-1}, 2, \{1\}^{k_2}).$$

Therefore, by combining Lemma 4.1 (59) and the equality (118) which has obtained just before, we have the explicit value of $\mathfrak{L}_{\mathcal{A},\{1\}^{k_1+k_2}}^{\text{III},*}(\{1\}^{k_1-1}, 2, \{1\}^{k_2-1}, 2)$ if $k_1 + k_2$ is odd. By translating FSSMP into FHSMP, we have the equality (124) and the first value of (129). The equality (127) and the rest of (129) are obtained by Proposition 3.11. By Corollary 3.13, we have the following equality:

$$\mathfrak{L}_{\mathcal{A},(k_1,k_2,k_3)}^{\text{III},*}(-1, 1, 1) = -\mathfrak{L}_{\mathcal{A},(\{1\}^{k_1+k_2-1}, 2, \{1\}^{k_3-1})}^{\text{III},*}(\{1\}^{k_1-1}, 2, \{1\}^{k_2+k_3-1}).$$

After translating FSSMPs into FHSMPs, we have the equality (131) by Lemma 4.12 (102) when k_1 is even and $k_2 + k_3$ odd and we have the explicit value of $\mathfrak{L}_{\mathcal{A},(1,2)}^{*,*}(2, 1/2)$ by Lemma 4.12 (106) when $k_1 = k_2 = k_3 = 1$. The equality (133) and the explicit value of $\mathfrak{L}_{\mathcal{A},(2,1)}^{*,*}(2, 1/2)$ are obtained by Proposition 3.11. The star cases of the equality (139) are obtained by the following relations:

$$\mathfrak{L}_{\mathcal{A},2}(2)\mathfrak{L}_{\mathcal{A},1}(1/2) = \mathfrak{L}_{\mathcal{A},(2,1)}^{*,*}(2, 1/2) + \mathfrak{L}_{\mathcal{A},(1,2)}^{*,*}(1/2, 2) - \zeta_{\mathcal{A}}(3)$$

and

$$\mathfrak{L}_{\mathcal{A},1}(2)\mathfrak{L}_{\mathcal{A},2}(1/2) = \mathfrak{L}_{\mathcal{A},(1,2)}^{*,*}(2, 1/2) + \mathfrak{L}_{\mathcal{A},(2,1)}^{*,*}(1/2, 2) - \zeta_{\mathcal{A}}(3).$$

By Theorem 3.12, we have the following equality:

$$\mathfrak{L}_{\mathcal{A},(k_1,k_2,k_3)}^{\text{III},*}(-1, -1, 1) = -\mathfrak{L}_{\mathcal{A},\{1\}^{w'}}^{\text{III},*}(\{1\}^{k_1-1}, 2, \{1\}^{k_2-1}, 2, \{1\}^{k_3}).$$

After translating FSSMPs into FHSMPs, we have the equalities (135) and (137) by Lemma 4.12 (104) when k_1 is even, k_2 odd, and k_3 even and we have the explicit value of $\mathfrak{L}_{\mathcal{A},\{1\}^3}^{*,*}(2, 1, 1/2)$ by Lemma 4.12 (107) when $k_1 = k_2 = k_3 = 1$.

Next, we prove non-star cases. By Theorem 3.15 (41), we have

$$(142) \quad \begin{aligned} & (-1)^w \mathfrak{L}_{\mathcal{A},\{1\}^w}^*(\{1\}^{k_1-1}, 1/2, 2, \{1\}^{k_2-1}) \\ &= -\mathfrak{L}_{\mathcal{A},\{1\}^w}^{*,*}(\{1\}^{k_2-1}, 2, 1/2, \{1\}^{k_1-1}) - (-1)^{k_1} \tilde{\mathfrak{L}}_{\mathcal{A},\{1\}^{k_1}}(1/2) \tilde{\mathfrak{L}}_{\mathcal{A},\{1\}^{k_2}}^*(2). \end{aligned}$$

Suppose $k_1, k_2 \geq 2$. By Proposition 4.3 (67) and (68), we have

$$\tilde{\mathfrak{L}}_{\mathcal{A},\{1\}^{k_1}}(1/2) \tilde{\mathfrak{L}}_{\mathcal{A},\{1\}^{k_2}}^*(2) = -\frac{(2^{k_1-1}-1)(2^{k_2-1}-1)}{2^{w-3}k_1k_2} B_{\mathbf{p}-k_1} B_{\mathbf{p}-k_2}.$$

If w is odd, then

$$\mathfrak{L}_{\mathcal{A},\{1\}^w}^*(\{1\}^{k_1-1}, 1/2, 2, \{1\}^{k_2-1}) = \mathfrak{L}_{\mathcal{A},\{1\}^w}^{*,*}(\{1\}^{k_2-1}, 2, 1/2, \{1\}^{k_1-1})$$

holds and if w is even, then

$$\begin{aligned} & \mathfrak{L}_{\mathcal{A},\{1\}^w}^*(\{1\}^{k_1-1}, 1/2, 2, \{1\}^{k_2-1}) \\ &= \frac{(2^{k_1-1}-1)(2^{k_2-1}-1)}{2^{w-3}k_1k_2} B_{\mathbf{p}-k_1} B_{\mathbf{p}-k_2} + (-1)^{k_1} \frac{(2^{k_1-1}-1)(2^{k_2-1}-1)}{2^{w-3}k_1k_2} B_{\mathbf{p}-k_1} B_{\mathbf{p}-k_2} = 0, \end{aligned}$$

holds by the equality (120). The case $k_1 = 1$ and $k_2 = 1$ are similar. Therefore we obtain the equalities (117) and (119). Suppose that w is odd. By Theorem 3.15 (41), we have

$$(143) \quad \begin{aligned} & \mathfrak{L}_{\mathcal{A},\{1\}^w}^*(2, \{1\}^{k_1-2}, 1/2, 2, \{1\}^{k_2-1}) = \mathfrak{L}_{\mathcal{A},\{1\}^w}^{*,*}(\{1\}^{k_2-1}, 2, 1/2, \{1\}^{k_1-2}, 2) \\ &+ \sum_{j=1}^{k_1-1} (-1)^j \mathfrak{L}_{\mathcal{A},\{1\}^j}^*(2) \mathfrak{L}_{\mathcal{A},\{1\}^{w-j}}^{*,*}(\{1\}^{k_2-1}, 2, 1/2, \{1\}^{k_1-j-1}) \\ &+ (-1)^{k_1} \mathfrak{L}_{\mathcal{A},\{1\}^{k_1}}^*(2, \{1\}^{k_1-2}, 1/2) \tilde{\mathfrak{L}}_{\mathcal{A},\{1\}^{k_2}}^*(2). \end{aligned}$$

Suppose that k_2 is even. $\tilde{\mathfrak{L}}_{\mathcal{A},\{1\}^{k_2}}^*(2) = 0$ holds by Proposition 4.3 (67). Furthermore, if j is even, $\mathfrak{L}_{\mathcal{A},\{1\}^j}^*(2) = 0$ holds and if j is odd, $\mathfrak{L}_{\mathcal{A},\{1\}^{w-j}}^{*,*}(\{1\}^{k_2-1}, 2, 1/2, \{1\}^{k_1-j-1}) = 0$ holds by the equality (120). Hence,

$$\mathfrak{L}_{\mathcal{A},\{1\}^w}^*(2, \{1\}^{k_1-2}, 1/2, 2, \{1\}^{k_2-1}) = \mathfrak{L}_{\mathcal{A},\{1\}^w}^{*,*}(\{1\}^{k_2-1}, 2, 1/2, \{1\}^{k_1-2}, 2)$$

holds and we have the equality (122) by the equality (124). If $m \geq 5$ is odd, we have

$$\begin{aligned} & \mathfrak{L}_{\mathcal{A},\{1\}^m}^*(2, 1/2, 2, \{1\}^{m-3}) - \mathfrak{L}_{\mathcal{A},\{1\}^m}^{*,*}(\{1\}^{m-3}, 2, 1/2, 2) \\ &= -\mathfrak{L}_{\mathcal{A},1}(2) \mathfrak{L}_{\mathcal{A},\{1\}^{m-1}}^{*,*}(\{1\}^{m-3}, 2, 1/2) + \mathfrak{L}_{\mathcal{A},\{1\}^2}^*(2, 1/2) \tilde{\mathfrak{L}}_{\mathcal{A},\{1\}^{m-2}}^*(2) \\ &= -(-2q_{\mathbf{p}}(2)) \frac{1-2^{m-3}}{2^{m-4}(m-2)} q_{\mathbf{p}}(2) B_{\mathbf{p}-m+2} + (-2q_{\mathbf{p}}(2))^2 \frac{1-2^{m-3}}{2^{m-4}} \frac{B_{\mathbf{p}-m+2}}{m-2} \\ &= 0 \end{aligned}$$

by the equality (143), Lemma 4.8 (80), the equality (121), Theorem 4.16 (147) below, and Proposition 4.3 (67). The case $m = 3$ is similar. Therefore, we have the equality (123). The equalities (125) and (126) are obtained by Proposition 3.11. The equalities (128) are obtained by Corollary 3.16 (42) and the equality (129). Since the equality (130) is obtained by Proposition 3.11, we prove the equality (132). Suppose that $k_1 + k_2$ is odd and k_3 even. By Theorem 3.15 (41), we have

$$\begin{aligned} \mathfrak{L}_{\mathcal{A},(\{1\}^{k_1-1},2,\{1\}^{k_2+k_3-1})}(\{1\}^{k_1+k_2-2},1/2,2,\{1\}^{k_3-1}) &= -\mathfrak{L}_{\mathcal{A},(\{1\}^{k_2+k_3-1},2,\{1\}^{k_1-1})}(\{1\}^{k_3-1},2,1/2,\{1\}^{k_1+k_2-2}) \\ &\quad - (-1)^{k_1+k_2-1} \tilde{\mathfrak{L}}_{\mathcal{A},(\{1\}^{k_1-1},2,\{1\}^{k_2-1})}(1/2)\tilde{\mathfrak{L}}_{\mathcal{A},\{1\}^{k_3}}^*(2). \end{aligned}$$

Since $\tilde{\mathfrak{L}}_{\mathcal{A},\{1\}^{k_3}}^*(2) = 0$ by Proposition 4.3 (67), we have the equality (132) by the equality (131). Suppose that k_1 is even, k_2 odd, k_3 even, and $k_2 > 1$. By Theorem 3.15 (41), we have

$$\begin{aligned} \mathfrak{L}_{\mathcal{A},\{1\}^{w'}}(\{1\}^{k_1-1},1/2,2,\{1\}^{k_2-2},1/2,2,\{1\}^{k_3-1}) &= \mathfrak{L}_{\mathcal{A},\{1\}^{w'}}(\{1\}^{k_3-1},2,1/2,\{1\}^{k_2-2},2,1/2,\{1\}^{k_1-1}) \\ &\quad + (-1)^{k_1} \tilde{\mathfrak{L}}_{\mathcal{A},\{1\}^{k_1}}(1/2)\mathfrak{L}_{\mathcal{A},\{1\}^{k_2+k_3}}^{*,*}(\{1\}^{k_3-1},2,1/2,\{1\}^{k_2-2},2) \\ &\quad + \sum_{j=0}^{k_2-2} (-1)^{k_1+j+1} \mathfrak{L}_{\mathcal{A},\{1\}^{k_1+j+1}}(\{1\}^{k_1-1},1/2,2,\{1\}^j) \mathfrak{L}_{\mathcal{A},\{1\}^{k_2+k_3-j-1}}^{*,*}(\{1\}^{k_3-1},2,1/2,\{1\}^{k_2-j-2}) \\ &\quad + (-1)^{k_1+k_2} \mathfrak{L}_{\mathcal{A},\{1\}^{k_1+k_2}}(\{1\}^{k_1-1},1/2,2,\{1\}^{k_2-2},1/2) \tilde{\mathfrak{L}}_{\mathcal{A},\{1\}^{k_3}}^*(2). \end{aligned}$$

For $j = 0, \dots, k_2 - 2$, if j is odd, $\mathfrak{L}_{\mathcal{A},\{1\}^{k_1+j+1}}(\{1\}^{k_1-1},1/2,2,\{1\}^j) = 0$ holds by the equality (119) and if j is even,

$$\mathfrak{L}_{\mathcal{A},\{1\}^{k_2+k_3-j-1}}^{*,*}(\{1\}^{k_3-1},2,1/2,\{1\}^{k_2-j-2}) = (\text{a certain element of } \mathcal{A}) \times B_{\mathbf{p}-k_3} = 0$$

holds by the equality (120). Furthermore, $\tilde{\mathfrak{L}}_{\mathcal{A},\{1\}^{k_1}}(1/2) = \tilde{\mathfrak{L}}_{\mathcal{A},\{1\}^{k_3}}^*(2) = 0$ holds by Proposition 4.3 (67) and (68). Therefore we obtain the equality (134) by the equality (135). Similarly, we see that the equality (136) holds. The non-star cases of the equalities (138), (139), and (140) are also obtained by the star cases of them. \square

Remark 4.15. The case $m = 3$ of the equalities (128)

$$(144) \quad \mathfrak{L}_{\mathcal{A},\{1\}^3}^*(1,2,1) = -2\mathfrak{L}_{\mathcal{A},\{1\}^3}^*(1,1/2,1) = \frac{1}{2}B_{\mathbf{p}-3}$$

also have been obtained by Tauraso and J. Zhao [28, Proposition 7.1 (85)].

Theorem 4.16. *Let k_1 and k_2 be positive even integers and $w = k_1 + k_2 + 1$. Then we have*

$$(145) \quad \mathfrak{L}_{\mathcal{A}_2, \{1\}^{w-1}}^* (\{1\}^{k_1-1}, 1/2, 2, \{1\}^{k_2-1}) = -\frac{1}{2} \left\{ 1 + \frac{2^{w-1} - 1}{2^{w-1}} \binom{w}{k_2} \right\} \frac{B_{\mathbf{p}-w}}{w} \mathbf{p},$$

$$(146) \quad \mathfrak{L}_{\mathcal{A}_2, \{1\}^{w-1}}^{*,*} (\{1\}^{k_1-1}, 2, 1/2, \{1\}^{k_2-1}) = \frac{1}{2} \left\{ 1 + \frac{2^{w-1} - 1}{2^{w-1}} \binom{w}{k_1} \right\} \frac{B_{\mathbf{p}-w}}{w} \mathbf{p},$$

$$(147) \quad \mathfrak{L}_{\mathcal{A}_2, \{1\}^2}^* (2, 1/2) = -2q_{\mathbf{p}}(2)^2 + \left(q_{\mathbf{p}}(2)^3 - \frac{7}{8} B_{\mathbf{p}-3} \right) \mathbf{p},$$

$$(148) \quad \mathfrak{L}_{\mathcal{A}_2, \{1\}^2}^{*,*} (2, 1/2) = -2q_{\mathbf{p}}(2)^2 + \left(q_{\mathbf{p}}(2)^3 - \frac{5}{24} B_{\mathbf{p}-3} \right) \mathbf{p},$$

$$(149) \quad \mathfrak{L}_{\mathcal{A}_2, \{1\}^2}^* (1/2, 2) = \frac{5}{24} B_{\mathbf{p}-3} \mathbf{p},$$

$$(150) \quad \mathfrak{L}_{\mathcal{A}_2, \{1\}^2}^{*,*} (1/2, 2) = \frac{7}{8} B_{\mathbf{p}-3} \mathbf{p}.$$

Proof. By Theorem 3.12, we have

$$\begin{aligned} & \mathfrak{L}_{\mathcal{A}_2, (k_1, k_2)}^{*,*} (-1, -1) + \left(\mathfrak{L}_{\mathcal{A}_2, (1, k_1, k_2)}^{*,*} (1, -1, -1) - \mathfrak{L}_{\mathcal{A}_2, (k_1+1, k_2)}^{*,*} (-1, -1) \right) \mathbf{p} \\ &= -\mathfrak{L}_{\mathcal{A}_2, \{1\}^{k_1+k_2}}^{*,*} (\{1\}^{k_1-1}, 2, 1/2, \{1\}^{k_2-1}). \end{aligned}$$

Therefore the equality (146) is obtained by Lemma 4.12 (99), (103), and Lemma 4.13 (112). By Theorem 3.15 (41), we have

$$\begin{aligned} \mathfrak{L}_{\mathcal{A}_2, \{1\}^{w-1}}^* (\{1\}^{k_1-1}, 1/2, 2, \{1\}^{k_2-1}) &= -\mathfrak{L}_{\mathcal{A}_2, \{1\}^{w-1}}^{*,*} (\{1\}^{k_2-1}, 2, 1/2, \{1\}^{k_1-1}) \\ &\quad - \sum_{j=1}^{k_1-1} (-1)^j \zeta_{\mathcal{A}_2}(\{1\}^j) \mathfrak{L}_{\mathcal{A}_2, \{1\}^{w-j-1}}^{*,*} (\{1\}^{k_2-1}, 2, 1/2, \{1\}^{k_1-j-1}) \\ &\quad - (-1)^{k_1} \widetilde{\mathfrak{L}}_{\mathcal{A}_2, \{1\}^{k_1}} (1/2) \widetilde{\mathfrak{L}}_{\mathcal{A}_2, \{1\}^{k_2}}^* (2) \\ &\quad - \sum_{i=0}^{k_2-2} (-1)^{k_1+i+1} \mathfrak{L}_{\mathcal{A}_2, \{1\}^{k_1+i+1}}^* (\{1\}^{k_1-1}, 1/2, 2, \{1\}^i) \zeta_{\mathcal{A}_2}^* (\{1\}^{k_2-i-1}). \end{aligned}$$

For $j = 1, \dots, k_1 - 1$, if j is odd, then $\zeta_{\mathcal{A}_2}(\{1\}^j) = 0$ and if j is even, then

$$\zeta_{\mathcal{A}_2}(\{1\}^j) \mathfrak{L}_{\mathcal{A}_2, \{1\}^{w-j-1}}^{*,*} (\{1\}^{k_2-1}, 2, 1/2, \{1\}^{k_1-j-1}) = (\text{a certain element of } \mathcal{A}_2) \times B_{\mathbf{p}-k_2} = 0$$

by Proposition 3.7 (23) and Theorem 4.14 (120). By Theorem 4.6, we have

$$\widetilde{\mathfrak{L}}_{\mathcal{A}_2, \{1\}^{k_1}} (1/2) \widetilde{\mathfrak{L}}_{\mathcal{A}_2, \{1\}^{k_2}}^* (2) = 0.$$

For $i = 0, \dots, k_2 - 2$, if i is even, then $\zeta_{\mathcal{A}_2}^*(\{1\}^{k_2-i-1}) = 0$ holds and if i is odd, then

$$\mathfrak{L}_{\mathcal{A}_2, \{1\}^{k_1+i+1}}^* (\{1\}^{k_1-1}, 1/2, 2, \{1\}^i) \zeta_{\mathcal{A}_2}^* (\{1\}^{k_2-i-1}) = 0$$

by Proposition 3.7 (23) and Theorem 4.14 (119). Therefore, we have the equality (145) by the equality (146). We also have the equality (148) by applying Theorem 3.12. The

equalities (147), (149), and (150) are obtained by the following relations:

$$\begin{aligned}\mathfrak{L}_{\mathcal{A}_2, \{1\}^2}^{*,*}(2, 1/2) &= \mathfrak{L}_{\mathcal{A}_2, \{1\}^2}^*(2, 1/2) + \zeta_{\mathcal{A}_2}(2), \\ \mathfrak{L}_{\mathcal{A}_2, 1}(2) \mathfrak{L}_{\mathcal{A}_2, 1}(1/2) &= \mathfrak{L}_{\mathcal{A}_2, \{1\}^2}^*(2, 1/2) + \mathfrak{L}_{\mathcal{A}_2, \{1\}^2}^*(1/2, 2) + \zeta_{\mathcal{A}_2}(2), \\ \mathfrak{L}_{\mathcal{A}_2, 1}(2) \mathfrak{L}_{\mathcal{A}_2, 1}(1/2) &= \mathfrak{L}_{\mathcal{A}_2, \{1\}^2}^{*,*}(2, 1/2) + \mathfrak{L}_{\mathcal{A}_2, \{1\}^2}^{*,*}(1/2, 2) - \zeta_{\mathcal{A}_2}(2).\end{aligned}$$

□

4.2. Relation between Ono-Yamamoto's FMPs and our FMPs. Ono and Yamamoto gave another definition of finite multiple polylogarithms in the paper [21]. Their purpose is to establish the shuffle relation of FMPs ([21, Theorem 3.6]). In this subsection, we give the relation between Ono-Yamamoto's FMPs and our FMPs. Furthermore, we calculate some special values of Ono-Yamamoto's FMPs.

Definition 4.17. Let $\mathbb{k} = (k_1, \dots, k_m)$ be an index. Then *Ono-Yamamoto's finite multiple polylogarithm* $\text{li}_{\mathbb{k}}(t) \in \mathcal{A}_{\mathbb{Z}[t]}$ is defined by

$$\text{li}_{\mathbb{k}}(t) := \left(\sum'_{0 < l_1, \dots, l_m < p} \frac{t^{l_1 + \dots + l_m}}{l_1^{k_1} (l_1 + l_2)^{k_2} \dots (l_1 + \dots + l_m)^{k_m}} \right)_p,$$

where the summation \sum' runs over only fractions whose denominators are prime to p .

By the substitution $l_i \mapsto p - l_i$, we have

$$(151) \quad \text{li}_{\mathbb{k}}(r) = (-1)^{\text{wt}(\mathbb{k})} r^{\text{dep}(\mathbb{k})} \text{li}_{\mathbb{k}}(r^{-1})$$

for any non-zero rational number r .

We prepare the following notations to discuss the relation between Ono-Yamamoto's FMPs and our FMPs (cf. [21, Section 2]):

$$[l] := \{1, \dots, l\},$$

$$\Phi_{m,l} := \{\phi: [m] \rightarrow [l] : \text{surjective} \mid \phi(a) \neq \phi(a+1) \text{ for all } a \in [m-1]\},$$

$$m_\phi := l \text{ when } \phi \in \Phi_{m,l},$$

$$\Phi_m := \bigsqcup_{l=1}^m \Phi_{m,l}, \quad \delta_\phi(i) := \#\{a \in [i-1] \mid \phi(a) > \phi(a+1)\} \text{ for } \phi \text{ in } \Phi_m,$$

$$\beta: \Phi_m \rightarrow [m] \text{ is defined by } \beta(\phi) := \delta_\phi(m) + 1, \quad \Phi_m^i := \beta^{-1}(i),$$

where i, l , and m are positive integers.

Proposition 4.18. *Let $\mathbb{k} = (k_1, \dots, k_m)$ be an index. Then we have*

$$(152) \quad \text{li}_{\mathbb{k}}(t) = \sum_{i=1}^m t^{(i-1)\mathbf{p}} \sum_{\phi \in \Phi_m^i} \mathfrak{L}_{\mathcal{A}, (\sum_{\phi(j)=m_\phi} k_j, \dots, \sum_{\phi(j)=1} k_j)}^* (\{1\}^{m_\phi - \phi(m)}, t, \{1\}^{\phi(m)-1}).$$

Proof. We omit the proof because it is completely the same as the proof of [21, Proposition 2.4]. □

Corollary 4.19. *Let k_1, k_2 , and k_3 be positive integers. Then we have*

$$(153) \quad \text{li}_{(k_1, k_2)}(t) = \mathcal{L}_{\mathcal{A}, (k_2, k_1)}(t) + t^{\mathbf{p}} \widetilde{\mathcal{L}}_{\mathcal{A}, (k_1, k_2)}(t),$$

$$(154) \quad \begin{aligned} \text{li}_{(k_1, k_2, k_3)}(t) &= \mathcal{L}_{\mathcal{A}, (k_3, k_2, k_1)}(t) + t^{\mathbf{p}} \mathcal{L}_{\mathcal{A}, (k_3, k_1, k_2)}(t) + t^{\mathbf{p}} \widetilde{\mathcal{L}}_{\mathcal{A}, (k_2, k_1, k_3)}(t) \\ &\quad + t^{\mathbf{p}} \mathcal{L}_{\mathcal{A}, (k_2, k_3, k_1)}^*(1, t, 1) + t^{\mathbf{p}} \mathcal{L}_{\mathcal{A}, (k_1, k_3, k_2)}^*(1, t, 1) \\ &\quad + t^{\mathbf{p}} \mathcal{L}_{\mathcal{A}, (k_1 + k_3, k_2)}(t) + t^{\mathbf{p}} \widetilde{\mathcal{L}}_{\mathcal{A}, (k_2, k_1 + k_3)}(t) + t^{2\mathbf{p}} \widetilde{\mathcal{L}}_{\mathcal{A}, (k_1, k_2, k_3)}(t). \end{aligned}$$

In general, it is difficult to calculate each term in the right hand side of (152) for $1 < i < m$. Therefore it seems to hard to calculate special values of Ono-Yamamoto's FMPs. However, we can evaluate the following values:

Proposition 4.20. *Let $\mathbb{k} = (k_1, \dots, k_m)$ be an index. Then we have*

$$(155) \quad \text{li}_{\mathbb{k}}(1) = 0,$$

$$(156) \quad \text{li}_{\{1\}^2}(-1) = \text{li}_{\{1\}^2}(2) = 2q_{\mathbf{p}}(2)^2, \quad \text{li}_{\{1\}^2}(1/2) = \frac{1}{2}q_{\mathbf{p}}(2)^2,$$

$$(157) \quad \text{li}_{(1,2)}(-1) = \text{li}_{(2,1)}(-1) = 0,$$

$$(158) \quad \text{li}_{(1,2)}(2) = \frac{2}{3}q_{\mathbf{p}}(2)^3 - \frac{2}{3}B_{\mathbf{p}-3}, \quad \text{li}_{(2,1)}(2) = \frac{2}{3}q_{\mathbf{p}}(2)^3 + \frac{4}{3}B_{\mathbf{p}-3},$$

$$(159) \quad \text{li}_{(1,2)}(1/2) = -\frac{1}{6}q_{\mathbf{p}}(2)^3 + \frac{1}{6}B_{\mathbf{p}-3}, \quad \text{li}_{(2,1)}(1/2) = -\frac{1}{6}q_{\mathbf{p}}(2)^3 - \frac{1}{3}B_{\mathbf{p}-3},$$

$$(160) \quad \text{li}_{\{1\}^3}(-1) = \text{li}_{\{1\}^3}(2) = -\frac{4}{3}q_{\mathbf{p}}(2)^3 - \frac{2}{3}B_{\mathbf{p}-3}, \quad \text{li}_{\{1\}^3}(1/2) = \frac{1}{6}q_{\mathbf{p}}(2)^3 + \frac{1}{12}B_{\mathbf{p}-3}.$$

Proof. If $m = 1$, we have $\text{li}_{\mathbb{k}}(1) = \zeta_{\mathcal{A}}(\mathbb{k}) = 0$. We assume that m is greater than or equal to 2. Let l be one of $2, \dots, m$ and S_l the l -th symmetric group. We define an equivalence relation on $\Phi_{m,l}$ as follows: $\phi \sim \phi'$ holds for $\phi, \phi' \in \Phi_{m,l}$ if and only if there exists $\sigma \in S_l$ such that $\phi = \sigma \circ \phi'$ holds. We take and fix a system of representatives $\{\phi_{l,1}, \dots, \phi_{l,i_l}\}$ of the quotient set $\Phi_{m,l}/S_l$ where i_l is the cardinality of $\Phi_{m,l}/S_l$. Then, by Proposition 4.18, we have

$$\begin{aligned} \text{li}_{\mathbb{k}}(1) &= \sum_{\phi \in \Phi_m} \zeta_{\mathcal{A}} \left(\sum_{\phi(j)=m_{\phi}} k_j, \dots, \sum_{\phi(j)=1} k_j \right) = \sum_{l=2}^m \sum_{\phi \in \Phi_{m,l}} \zeta_{\mathcal{A}} \left(\sum_{\phi(j)=l} k_j, \dots, \sum_{\phi(j)=1} k_j \right) \\ &= \sum_{l=2}^m \sum_{s=1}^{i_l} \left(\sum_{\sigma \in S_l} \zeta_{\mathcal{A}} \left(\sigma \left(\sum_{\phi_{l,s}(j)=l} k_j, \dots, \sum_{\phi_{l,s}(j)=1} k_j \right) \right) \right). \end{aligned}$$

We see that this is zero by Proposition 3.7 (30).

The equalities (156), (157), (158), (159), and (160) are obtained by Corollary 4.19, Lemma 4.1, Proposition 4.3, Proposition 4.9, Lemma 4.12 (108), and Remark 4.15 (144). \square

APPENDIX A. AN ELEMENTARY PROOF OF THE CONGRUENCES BETWEEN
BERNOULLI NUMBERS AND GENERALIZED BERNOULLI NUMBERS

Let N be a positive integer and χ a Dirichlet character modulo N . The generalized Bernoulli numbers $B_{n,\chi}$ are defined by

$$\sum_{a=1}^N \frac{\chi(a)te^{at}}{e^{Nt} - 1} = \sum_{n=0}^{\infty} B_{n,\chi} \frac{t^n}{n!}.$$

We use the following formula for non-trivial character χ :

$$(161) \quad B_{1,\chi} = \frac{1}{N} \sum_{a=1}^N \chi(a)a.$$

In this appendix, we give an elementary proof of the following classical congruence:

Proposition A.1. *Let k be an integer greater than or equal to 2 and p a prime greater than $k + 1$. Let ω_p be the Teichmüller character. Then*

$$(162) \quad B_{1,\omega_p^{-k}} \equiv -\frac{B_{p-k}}{k} \pmod{p}.$$

For an integer m and a positive integer n , we define B_{1,ω_p^m} to be $(B_{1,\omega_p^m} \bmod p^n) \in \mathcal{A}_n$. Then we have $B_{1,\omega_p^{-k}} = \widehat{B}_{\mathbf{p}-k}$ in \mathcal{A} by the above proposition.¹ Therefore we can express some FMZVs or some special values of FMPs by the generalized Bernoulli numbers, e.g.

$$\zeta_{\mathcal{A}}(k_1, k_2) = (-1)^{k_1-1} \binom{k_1 + k_2}{k_1} B_{1,\omega_p^{-k_1-k_2}}$$

where k_1 and k_2 are positive integers.

We can prove Proposition A.1 by using the Kubota-Leopoldt p -adic L -function (cf. [29, Corollary 5.15]), however, we give a proof of stronger results by elementary calculations.

Theorem A.2. *Let k be an integer greater than or equal to 2. If k is odd, then*

$$(163) \quad B_{1,\omega_p^{-k}} = \frac{k^2 - k + 6}{2} \widehat{B}_{\mathbf{p}-k} - (k^2 - 2k + 3) \widehat{B}_{2\mathbf{p}-k-1} + \frac{k^2 - 3k + 2}{2} \widehat{B}_{3\mathbf{p}-k-2} \\ + k(\widehat{B}_{\mathbf{p}-k} - \widehat{B}_{2\mathbf{p}-k-1})\mathbf{p} - k\widehat{B}_{\mathbf{p}-k-2}\mathbf{p}^2 \text{ in } \mathcal{A}_3$$

and if k is even, then

$$(164) \quad B_{1,\omega_p^{-k}} = -3k(\widehat{B}_{\mathbf{p}-k-1} - \widehat{B}_{2\mathbf{p}-k-2})\mathbf{p} \text{ in } \mathcal{A}_3.$$

Corollary A.3. *Let k be an integer greater than or equal to 2. Then*

$$(165) \quad B_{1,\omega_p^{-k}} = (k+2)\widehat{B}_{\mathbf{p}-k} - (k+1)\widehat{B}_{2\mathbf{p}-k-1} \text{ in } \mathcal{A}_2,$$

$$(166) \quad B_{1,\omega_p^{-k}} = \widehat{B}_{\mathbf{p}-k} \text{ in } \mathcal{A}.$$

¹See Example 3.3 about the notation \widehat{B}_m . The element $\widehat{B}_{\mathbf{p}-k}$ is considered as a finite analogue of Riemann zeta value $\zeta(k)$. See [10].

Proof. We can prove these by the same method as the proof of Theorem A.2 or by Kummer's congruences below. \square

We consider congruences in the p -adic integer ring \mathbb{Z}_p . First, we recall Kummer's congruences in the sense of Z. H. Sun.

Proposition A.4 (Z. H. Sun [23]). *Let p be an odd prime number. Let m and l be positive integers satisfying $l \not\equiv 0 \pmod{p-1}$. Then*

$$(167) \quad \widehat{B}_{m(p-1)+l} \equiv m\widehat{B}_{p-1+l} - (m-1)(1-p^{l-1})\widehat{B}_l \pmod{p^2}.$$

This congruence is a generalization of Kummer's congruence, that is, $\widehat{B}_n \equiv \widehat{B}_m \pmod{p}$ when $n \equiv m \pmod{p-1}$.

From now on, we fix an integer k greater than or equal to 2 and an odd prime p satisfying $p > k+4$.

Lemma A.5. *Let p and k be as above. Then*

$$\begin{aligned} \sum_{a=1}^{p-1} a^{1-k} &\equiv \begin{cases} -(k-1)(3\widehat{B}_{p-k} - 3\widehat{B}_{2p-k-1} + \widehat{B}_{3p-k-2})p - \binom{k+1}{3}\widehat{B}_{p-k-2}p^3 \pmod{p^4} & \text{if } k \text{ is odd,} \\ \binom{k}{2}(2\widehat{B}_{p-k-1} - \widehat{B}_{2p-k-2})p^2 \pmod{p^4} & \text{if } k \text{ is even,} \end{cases} \\ \sum_{a=1}^{p-1} a^{p-k} &\equiv \begin{cases} -k\widehat{B}_{p-k}p + \widehat{B}_{p-k}p^2 - \binom{k+2}{3}\widehat{B}_{p-k-2}p^3 \pmod{p^4} & \text{if } k \text{ is odd,} \\ \binom{k+1}{2}\widehat{B}_{p-k-1}p^2 - \frac{2k+1}{2}\widehat{B}_{p-k-1}p^3 \pmod{p^4} & \text{if } k \text{ is even,} \end{cases} \\ \sum_{a=1}^{p-1} a^{2p-k-1} &\equiv \begin{cases} -(k+1)\widehat{B}_{2p-k-1}p + 2\widehat{B}_{2p-k-1}p^2 - \binom{k+3}{3}\widehat{B}_{p-k-2}p^3 \pmod{p^4} & \text{if } k \text{ is odd,} \\ \binom{k+2}{2}\widehat{B}_{2p-k-2}p^2 - (2k+3)\widehat{B}_{p-k-1}p^3 \pmod{p^4} & \text{if } k \text{ is even,} \end{cases} \\ \sum_{a=1}^{p-1} a^{3p-k-2} &\equiv \begin{cases} -(k+2)\widehat{B}_{3p-k-2}p + 3(2\widehat{B}_{2p-k-1} - \widehat{B}_{p-k})p^2 - \binom{k+4}{3}\widehat{B}_{p-k-2}p^3 \pmod{p^4} & \text{if } k \text{ is odd,} \\ \binom{k+3}{2}(2\widehat{B}_{2p-k-2} - \widehat{B}_{p-k-1})p^2 - \frac{3}{2}(2k+5)\widehat{B}_{p-k-1}p^3 \pmod{p^4} & \text{if } k \text{ is even.} \end{cases} \end{aligned}$$

Proof. The first congruence is deduced by Proposition 3.7 (25). Let m be an integer greater than 3. Then, by Faulhaber's formula, we have

$$\begin{aligned} \sum_{a=1}^{p-1} a^m &= \frac{1}{m+1} \sum_{i=1}^{m+1} \binom{m+1}{i} B_{m+1-i} p^i \\ &\equiv pB_m + \frac{p^2}{2}mB_{m-1} + \frac{p^3}{6}m(m-1)B_{m-2} + \frac{p^4}{24}m(m-1)(m-2)B_{m-3} \pmod{p^4} \end{aligned}$$

since $\frac{1}{m+1} \binom{m+1}{i} B_{m+1-i} p^i = p^4 \binom{m}{i-1} p B_{m+1-i} \frac{p^{i-5}}{i}$ and $pB_{m+1-i}, p^{i-5}/i$ are p -adic integers for $i \geq 5$. If $m = p-k, 2p-k-1$, or $3p-k-2$, then $\frac{p^4}{24}m(m-1)(m-2)B_{m-3}$ also vanishes mod p^4 since B_{m-3} is a p -adic integer by the von Staudt-Clausen theorem. Since $B_m = 0$ if $m \geq 3$ is odd, we have

$$\begin{aligned} \sum_{a=1}^{p-1} a^{p-k} &\equiv \begin{cases} pB_{p-k} + \frac{p^3}{6}k(k+1)B_{p-k-2} \pmod{p^4} & \text{if } k \text{ is odd,} \\ \frac{p^2}{2}(p-k)B_{p-k-1} \pmod{p^4} & \text{if } k \text{ is even,} \end{cases} \\ \sum_{a=1}^{p-1} a^{2p-k-1} &\equiv \begin{cases} pB_{2p-k-1} + \frac{p^3}{6}(k+1)(k+2)B_{2p-k-3} \pmod{p^4} & \text{if } k \text{ is odd,} \\ \frac{p^2}{2}(2p-k-1)B_{2p-k-2} \pmod{p^4} & \text{if } k \text{ is even,} \end{cases} \\ \sum_{a=1}^{p-1} a^{3p-k-2} &\equiv \begin{cases} pB_{3p-k-2} + \frac{p^3}{6}(k+2)(k+3)B_{3p-k-4} \pmod{p^4} & \text{if } k \text{ is odd,} \\ \frac{p^2}{2}(3p-k-2)B_{3p-k-3} \pmod{p^4} & \text{if } k \text{ is even.} \end{cases} \end{aligned}$$

We obtain the desired formulas by changing each B_m to \widehat{B}_m and by Kummer's congruence (167), e.g.

$$\widehat{B}_{3p-k-\varepsilon-2} \equiv 2\widehat{B}_{2p-k-\varepsilon-1} - \widehat{B}_{p-k-\varepsilon} \pmod{p^2}$$

where $\varepsilon \in \{0, 1\}$. □

Proof of Theorem A.2. We put $\omega := \omega_p$. For $a \in \mathbb{Z}_p^\times$, we define $\langle a \rangle$ to be $\omega(a)^{-1}a$. Then $\langle a \rangle \in 1 + p\mathbb{Z}_p$. Let

$$\log_p: \mathbb{Z}_p^\times \rightarrow p\mathbb{Z}_p$$

be a p -adic logarithm function and let \exp be the p -adic exponential function, which is the right inverse of \log_p . Then we have

$$(168) \quad \langle a \rangle = \exp(\log_p \langle a \rangle) = \exp(\log_p a) = \exp\left(\frac{1}{p-1} \log_p(a^{p-1})\right).$$

Let $\alpha := \frac{1}{p-1} \log_p(a^{p-1}) = \frac{1}{p-1} \log_p(1 + pq_p(a)) \in p\mathbb{Z}_p$ where $q_p(a)$ is the Fermat quotient (See Definition 4.5). Since we can check easily that $\alpha^n/n! \equiv 0 \pmod{p^4}$ if $n \geq 4$, we have

$$(169) \quad \langle a \rangle \equiv 1 + \alpha + \frac{\alpha^2}{2} + \frac{\alpha^3}{6} \pmod{p^4}.$$

By definition of a p -adic logarithm, we have

$$\log_p(1 + pq_p(a)) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} p^n q_p(a)^n \equiv pq_p(a) - \frac{1}{2} p^2 q_p(a)^2 + \frac{1}{3} p^3 q_p(a)^3 \pmod{p^4}.$$

Therefore, by the congruence (169), we have

$$\begin{aligned} \langle a \rangle &\equiv 1 + \frac{1}{p-1} \left(pq_p(a) - \frac{1}{2} p^2 q_p(a)^2 + \frac{1}{3} p^3 q_p(a)^3 \right) + \frac{1}{2} \left(\frac{1}{p-1} \right)^2 \left(pq_p(a) - \frac{1}{2} p^2 q_p(a)^2 + \frac{1}{3} p^3 q_p(a)^3 \right)^2 \\ &\quad + \frac{1}{6} \left(\frac{1}{p-1} \right)^3 \left(pq_p(a) - \frac{1}{2} p^2 q_p(a)^2 + \frac{1}{3} p^3 q_p(a)^3 \right)^3 \\ &\equiv 1 - q_p(a)p - (q_p(a) - q_p(a)^2)p^2 - \left(q_p(a) - \frac{3}{2} q_p(a)^2 + q_p(a)^3 \right) p^3 \pmod{p^4}. \end{aligned}$$

Note that the last congruence is obtained by $\frac{1}{p-1} = -\sum_{i=0}^{\infty} p^i$, $\left(\frac{1}{p-1}\right)^2 = \sum_{i=1}^{\infty} i p^{i-1}$, and $\left(\frac{1}{p-1}\right)^3 = -\sum_{i=2}^{\infty} i(i-1) p^{i-2}$. By the binomial expansion, we have

$$\begin{aligned} \langle a \rangle^k &\equiv 1 + k \left\{ -q_p(a)p - (q_p(a) - q_p(a)^2)p^2 - \left(q_p(a) - \frac{3}{2} q_p(a)^2 + q_p(a)^3 \right) p^3 \right\} \\ &\quad + \binom{k}{2} \left\{ -q_p(a)p - (q_p(a) - q_p(a)^2)p^2 - \left(q_p(a) - \frac{3}{2} q_p(a)^2 + q_p(a)^3 \right) p^3 \right\}^2 \\ &\quad + \binom{k}{3} \left\{ -q_p(a)p - (q_p(a) - q_p(a)^2)p^2 - \left(q_p(a) - \frac{3}{2} q_p(a)^2 + q_p(a)^3 \right) p^3 \right\}^3 \\ &\equiv 1 - k q_p(a)p - \left\{ k q_p(a) - \binom{k+1}{2} q_p(a)^2 \right\} p^2 \\ &\quad - \left\{ k q_p(a) - \left(k^2 + \frac{k}{2} \right) q_p(a)^2 + \frac{k^3 + 3k^2 - 4k}{6} q_p(a)^3 \right\} p^3 \pmod{p^4} \end{aligned}$$

Hence, by the equality (161), we have

$$\begin{aligned}
pB_{1,\omega^{-k}} &= \sum_{a=1}^{p-1} \omega^{-k}(a)a = \sum_{a=1}^{p-1} a^{1-k} \langle a \rangle^k \\
&\equiv \sum_{a=1}^{p-1} a^{1-k} - kp \sum_{a=1}^{p-1} a^{1-k} q_p(a) - \left\{ k \sum_{a=1}^{p-1} a^{1-k} q_p(a) - \binom{k+1}{2} \sum_{a=1}^{p-1} a^{1-k} q_p(a)^2 \right\} p^2 \\
&\quad - \left\{ k \sum_{a=1}^{p-1} a^{1-k} q_p(a) - \left(k^2 + \frac{k}{2} \right) \sum_{a=1}^{p-1} a^{1-k} q_p(a)^2 + \frac{k^3 + 3k^2 - 4k}{6} \sum_{a=1}^{p-1} a^{1-k} q_p(a)^3 \right\} p^3 \pmod{p^4}.
\end{aligned}$$

Since

$$\begin{aligned}
p \sum_{a=1}^{p-1} a^{1-k} q_p(a) &= - \sum_{a=1}^{p-1} a^{1-k} + \sum_{a=1}^{p-1} a^{p-k}, \\
p^2 \sum_{a=1}^{p-1} a^{1-k} q_p(a)^2 &= \sum_{a=1}^{p-1} a^{1-k} - 2 \sum_{a=1}^{p-1} a^{p-k} + \sum_{a=1}^{p-1} a^{2p-k-1}, \\
p^3 \sum_{a=1}^{p-1} a^{1-k} q_p(a)^3 &= - \sum_{a=1}^{p-1} a^{1-k} + 3 \sum_{a=1}^{p-1} a^{p-k} - 3 \sum_{a=1}^{p-1} a^{2p-k-1} + \sum_{a=1}^{p-1} a^{3p-k-2},
\end{aligned}$$

we have

$$\begin{aligned}
pB_{1,\omega^{-k}} &\equiv \left\{ \frac{k^3 + 6k^2 + 5k + 6}{6} + \left(k^2 + \frac{3}{2}k \right) p + kp^2 \right\} \sum_{a=1}^{p-1} a^{1-k} - \left\{ \frac{k^3 + 5k^2}{2} + (k^2 + k)p + kp^2 \right\} \sum_{a=1}^{p-1} a^{p-k} \\
&\quad + \left\{ \frac{k^3 + 4k^2 - 3k}{2} + \left(k^2 + \frac{k}{2} \right) p \right\} \sum_{a=1}^{p-1} a^{2p-k-1} - \frac{k^3 + 3k^2 - 4k}{6} \sum_{a=1}^{p-1} a^{3p-k-1} \pmod{p^4}.
\end{aligned}$$

If k is odd, by Lemma A.5, we have

$$\begin{aligned}
pB_{1,\omega^{-k}} &\equiv \left\{ \frac{k^3 + 6k^2 + 5k + 6}{6} + \left(k^2 + \frac{3}{2}k \right) p + kp^2 \right\} \left\{ -(k-1)(3\widehat{B}_{p-k} - 3\widehat{B}_{2p-k-1} + \widehat{B}_{3p-k-2})p \right. \\
&\quad \left. - \binom{k+1}{3} \widehat{B}_{p-k-2} p^3 \right\} \\
&\quad - \left\{ \frac{k^3 + 5k^2}{2} + (k^2 + k)p + kp^2 \right\} \left\{ -k\widehat{B}_{p-k}p + \widehat{B}_{p-k}p^2 - \binom{k+2}{3} \widehat{B}_{p-k-2} p^3 \right\} \\
&\quad + \left\{ \frac{k^3 + 4k^2 - 3k}{2} + \left(k^2 + \frac{k}{2} \right) p \right\} \left\{ -(k+1)\widehat{B}_{2p-k-1}p + 2\widehat{B}_{2p-k-1}p^2 - \binom{k+3}{3} \widehat{B}_{p-k-2} p^3 \right\} \\
&\quad - \frac{k^3 + 3k^2 - 4k}{6} \left\{ -(k+2)\widehat{B}_{3p-k-2}p + 3(2\widehat{B}_{2p-k-1} - \widehat{B}_{p-k})p^2 - \binom{k+4}{3} \widehat{B}_{p-k-2} p^3 \right\} \\
&\equiv \left\{ \frac{k^2 - k + 6}{2} \widehat{B}_{p-k} - (k^2 - 2k + 3) \widehat{B}_{2p-k-1} + \frac{k^2 - 3k + 2}{2} \widehat{B}_{3p-k-2} \right\} p \\
&\quad + k(\widehat{B}_{p-k} - \widehat{B}_{2p-k-1})p^2 - k\widehat{B}_{p-k-2}p^3 \pmod{p^4}.
\end{aligned}$$

If k is even, then

$$\begin{aligned}
pB_{1,\omega^{-k}} &\equiv \left\{ \frac{k^3 + 6k^2 + 5k + 6}{6} + \left(k^2 + \frac{3}{2}k \right) p + kp^2 \right\} \left\{ \binom{k}{2} (2\widehat{B}_{p-k-1} - \widehat{B}_{2p-k-2})p^2 \right\} \\
&\quad - \left\{ \frac{k^3 + 5k^2}{2} + (k^2 + k)p + kp^2 \right\} \left\{ \binom{k+1}{2} \widehat{B}_{p-k-1}p^2 - \frac{2k+1}{2} \widehat{B}_{p-k-1}p^3 \right\}
\end{aligned}$$

$$\begin{aligned}
& + \left\{ \frac{k^3 + 4k^2 - 3k}{2} + \left(k^2 + \frac{k}{2} \right) p \right\} \left\{ \binom{k+2}{2} \widehat{B}_{2p-k-2} p^2 - (2k+3) \widehat{B}_{p-k-1} p^3 \right\} \\
& - \frac{k^3 + 3k^2 - 4k}{6} \left\{ \binom{k+3}{2} (2\widehat{B}_{2p-k-2} - \widehat{B}_{p-k-1}) p^2 - \frac{3}{2} (2k+5) \widehat{B}_{p-k-1} p^3 \right\} \\
& \equiv -3k(\widehat{B}_{p-k-1} - \widehat{B}_{2p-k-2}) p^2 \pmod{p^4}.
\end{aligned}$$

This completes the proof. \square

APPENDIX B. TABLE OF SUFFICIENT CONDITIONS FOR CONGRUENCES

In this appendix, we give a sufficient condition that the congruence obtained as each p -component of the special value of a FMP holds. First, we list the equalities whose p -component congruences hold for all prime numbers: Theorem 1.3 (1) and (2), Proposition 3.7 (31) and (32), Proposition 3.11 (34), (35), and (36), Theorem 3.12 (37), Corollary 3.13 (38), Remark 3.14 (39) and (40), Theorem 3.15 (41), Corollary 3.16 (43), (44), and (45), Proposition 3.21 (51), Proposition 4.2 (66), Subsection 4.2 (151), Proposition 4.18 (152), and Corollary 4.19 (153) and (154). Here, we understand non-star sum (e.g. the p -component of $\zeta_{\mathcal{A}}(\mathbb{k})$) as 0 if $p \leq \text{dep}(\mathbb{k})$.

TABLE 1. Lemma 3.4, Proposition 3.7, Corollary 3.16, Theorem 3.18, Lemma 3.20, Theorem 3.22, Corollary 3.23, Lemma 4.1, Lemma 4.2, Proposition 4.3, Theorem 4.6, Lemma 4.8, Proposition 4.9, Proposition 4.11, Lemma 4.12, Lemma 4.13, Theorem 4.14, Remark 4.15, Theorem 4.16, Proposition 4.20, Theorem A.2, and Corollary A.3

number	(20)	(23)	(24)	(25)
condition	$p > n$	$p > mk + 2$	$p > k + 3$	$p > k + 4$
number	(26)	(27) and (28)	(29)	(30)
condition	$p > k_1 + k_2 - 1$	$p > w + 1$	$p > w'$	$p > \text{wt}(\mathbb{k}) + 1$
number	(33)	(42)	(46) and (47)	(49), (50), (52) - (58)
condition	$p > k_2$	$p > mk + 1$	$p > k_1 + k_2 + 1$	$p > m + 1$
number	(59) and (60)	(61)	(62) - (65)	(67) and (68)
condition	$p > w + 1$	$p > m + 2$	$p > w + 2$	$p > m + 1$
number	(69) - (72)	(74) and (75)	(79) - (95)	(96) - (98)
condition	$p > w + 1$	$p > m + 2$	$p > 3$	$p > 5$
number	(99) and (100)	(101)	(102) - (104)	(105) - (108)
condition	$p > k_1 + k_2 + 1$	$p > m + 2$	$p > w + 1$	$p > 3$
number	(111) and (112)	(113) and (114)	(115) and (116)	(117)-(120), (122), (124), (125), (127)
condition	$p > k_1 + k_2 + 2$	$p > 3$	$p > 5$	$p > w + 1$
number	(121)	(123), (126), (128), (129)	(130) - (135)	(136) and (137)
condition	$p > m + 2$	$p > m + 1$	$p > w' + 1$	$p > w + 2$
number	(138) - (140), (144)	(145) and (146)	(147) - (150)	(155)
condition	$p > 3$	$p > w + 1$	$p > 3$	$p > \text{wt}(\mathbb{k}) + 1$
number	(156) - (160)	(163) and (164)	(165)	(166)
condition	$p > 3$	$p > k + 4$	$p > k + 3$	$p > k + 2$

For example, (23) in the above table means that

$$\zeta_{p-1}^*(\{k\}^m) := \sum_{p-1 \geq n_1 \geq \dots \geq n_m \geq 1} \frac{1}{n_1^k \cdots n_m^k} \equiv k \frac{B_{p-mk-1}}{mk+1} p \pmod{p^2}$$

holds for any positive integers m, k and any prime p satisfying $p > mk + 2$. The mod p version

$$\zeta_{p-1}^*(\{k\}^m) \equiv 0 \pmod{p}$$

holds for $p > mk + 1$ (cf. [8, Theorem 4.4]).

ACKNOWLEDGEMENTS

The authors would like to thank Prof. Masanobu Kaneko for helpful comments, correcting several historical mistakes, and giving them his manuscript for the finite multiple zeta values. They also thank their advisor Prof. Tadashi Ochiai and Sho Ogaki for reading the manuscript carefully.

REFERENCES

- [1] M. Chamberland, K. Dilcher, *Divisibility properties of a class of binomial sums*, J. Number Theory **120** (2006), no. 2, 349–371.
- [2] K. Dilcher, *Some q -series identities related to divisor functions*, Discrete Math. **145** (1995), no. 1–3, 83–93.
- [3] K. Dilcher, L. Skula, *The cube of the Fermat quotient*, Integers **6** (2006), A24, 12 pp.
- [4] P. Elbaz-Vincent, H. Gangl, *On poly(ana)logs I*, Compositio Math. **130** (2002), no. 2, 161–214.
- [5] L. Euler, *Demonstratio insignis theorematis numerici circa unicias potestatum binomialium*, Nova Acta Academiae Scientiarum Imperialis Petropolitinae **15**, (1799/1802), 33–43.
- [6] S. Bang, J. E. Dawson, A. N. 't Woord, O. P. Lossers, V. Hernandez, *Problems and Solutions: Solutions: A Reciprocal Summation Identity: 10490*, Amer. Math. Monthly **106** (1999), no. 6, 588–590.
- [7] M. Hoffman, *Quasi-shuffle products*, J. Algebraic Combin. **11** (2000), no. 1, 49–68.
- [8] M. Hoffman, *Quasi-symmetric functions and mod p multiple harmonic sums*, Kyushu J. Math. **69** (2015), 345–366.
- [9] K. Ihara, J. Kajikawa, Y. Ohno, J. Okuda, *Multiple zeta values vs. multiple zeta-star values*, J. Algebra **332** (2011), 187–208.
- [10] M. Kaneko, *Finite multiple zeta values*, RIMS Kôkyûroku Bessatsu (Japanese), to appear.
- [11] M. Kaneko, D. Zagier, *Finite multiple zeta values*, in preparation.
- [12] G. Kawashima, *A class of relations among multiple zeta values*, J. Number Theory **129** (2009), no. 4, 755–788.
- [13] G. Kawashima, T. Tanaka, *Newton series and extended derivation relations for multiple L -values*, preprint arXiv:0801.3062.
- [14] M. Kontsevich, *The $1\frac{1}{2}$ -logarithm*, Appendix to “On poly(ana)logs I” by Elbaz-Vincent, P., Gangl, H., Compositio Math. **130** (2002), no. 2, 211–214.
- [15] M. Kontsevich, *Holonomic \mathcal{D} -modules and positive characteristic*, Jpn. J. Math. **4** (2009), no. 1, 1–25.
- [16] R. Meštrović, *On the mod p^2 determination of $\sum_{k=1}^{p-1} H_k/(k \cdot 2^k)$: another proof of a conjecture by Sun*, Publ. Math. Debrecen **82** (2013), no. 1, 107–123.
- [17] R. Meštrović, *Congruences involving the Fermat quotient*, Czechoslovak Math. J. **63**(138) (2013), no. 4, 949–968.
- [18] S. Mattarei, R. Tauraso, *Congruences of multiple sums involving sequences invariant under the binomial transform*, J. Integer Seq., **13** (2010), no. 5, Article 10. 5. 1, 12 pp.
- [19] S. Mattarei, R. Tauraso, *Congruences for central binomial sums and finite polylogarithms*, J. Number Theory, **133** (2013), no. 1, 131–157.

- [20] Kh. H. Pilehrood, T. H. Pilehrood, R. Tauraso, *New properties of multiple harmonic sums modulo p and p -analogues of Leshchiner's series*, Trans. Amer. Math. Soc. **366** (2014), no. 6, 3131–3159.
- [21] M. Ono, S. Yamamoto, *Shuffle product of finite multiple polylogarithms*, preprint arXiv:1502.06693.
- [22] J. H. Silverman, *Wieferich's criterion and the abc-conjecture*, J. Number Theory **30** (1988), no. 2, 226–237.
- [23] Z. H. Sun, *Congruences concerning Bernoulli numbers and Bernoulli polynomials*, Discrete Appl. Math. **105** (2000), no. 1–3, 193–223.
- [24] Z. H. Sun, *Congruences involving Bernoulli and Euler numbers*, J. Number Theory **128** (2008), no. 2, 280–312.
- [25] Z. W. Sun, *Arithmetic theory of harmonic numbers*, Proc. Amer. Math. Soc. **140** (2012), no. 2, 415–428.
- [26] Z. W. Sun, L. L. Zhao, *Arithmetic theory of harmonic numbers (II)*, Colloq. Math. **130** (2013), no. 1, 67–78.
- [27] R. Tauraso, *Congruences involving alternating multiple harmonic sums*, Electron. J. Combin. **17** (2010), no. 1, Research Paper 16, 11 pp.
- [28] R. Tauraso, J. Zhao, *Congruences of alternating multiple harmonic sums*, J. Comb. Number Theory **2** (2010), no. 2, 129–159.
- [29] L. C. Washington, *Introduction to cyclotomic fields. Second edition*, Graduate Texts in Mathematics, **83**, Springer-Verlag, New York, 1997.
- [30] S. Yamamoto, *Interpolation of multiple zeta and zeta-star values*, J. Algebra, Volume **385** (2013), 102–114.
- [31] J. Zhao, *Wolstenholme type theorem for multiple harmonic sums*, Int. J. Number Theory **4** (2008), no. 1, 73–106.
- [32] L. L. Zhao, Z. W. Sun, *Some curious congruences modulo primes*, J. Number Theory **130** (2010), no. 4, 930–935.
- [33] S. A. Zlobin, *Generating functions for the values of a multiple zeta function*, Vestnik Moskov. Univ. Ser. I Mat. Mekh. 2005, no. 2, 55–59, **73**; translation in Moscow Univ. Math. Bull. **60** (2005), no. 2, 44–48.

DEPARTMENT OF MATHEMATICS, GRADUATE SCHOOL OF SCIENCE OSAKA UNIVERSITY TOYONAKA, OSAKA 560-0043 JAPAN

E-mail address: k-sakugawa@cr.math.sci.osaka-u.ac.jp

DEPARTMENT OF MATHEMATICS, GRADUATE SCHOOL OF SCIENCE OSAKA UNIVERSITY TOYONAKA, OSAKA 560-0043 JAPAN

E-mail address: shinchan.prime@gmail.com